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ON THE CONCEPTS OF A PROBLEM AND PROBLEM-SOLVING*

RICHARD BELLMAN, The RAND Corporation, AND PAUL BROCK, Hughes Aircraft Company

1. Introduction. The past decade has seen startling changes in science and technology, much of it directly attributable to the use of large-scale computing devices. The power and versatility of these new devices have resolved many problems not only previously unsolved, but even previously unformulated. Yet the application of these new devices has barely begun. Now that overwhelming evidence of the effectiveness of these machines has accumulated, it is essential that serious and prolonged thought be devoted to the optimal utilization of current computers and to the future development of computers. It is clear that the field of mathematics is vitally affected by these new developments. It is worthwhile, then, to subject to critical analysis the concept of a *solution*—the primary objective of a mathematical theory—and, in addition, the even more fundamental concept of a *problem*.

In this paper, we wish in an informal fashion to discuss these matters. Our aim is to examine the role of the mathematician in providing a path from problem to solution. To do this, we shall discuss some of the different kinds of mathematical difficulties, the various ways in which computers can be used to resolve these difficulties, and a number of quite simply stated questions that give rise to these difficulties.

2. Problems. Problems may be divided roughly into three types. These are the *natural* problems arising in the world about us; the *model* problems, which we immediately recognize as an element of a class of similar problems; and, finally, the *symbolic* problems where an analytic formulation already exists. To solve a problem by mathematical techniques, we transform it step-by-step from a natural problem to a model problem and from a model problem to a symbolic problem. Obtaining the solution of the symbolic problem in analytic terms, we must interpret it suitably to obtain a solution of the model problem and then go from here to the natural problem. None of these steps are trivial and each requires experience and skill.

3. Solutions. In the course of this discussion, we introduced the concept of a *solution*, a term we shall employ on an intuitive basis. It is clear that an action, an effect which is equivalent to this solution, must bear favorable comparison with other possible actions, and that, furthermore, it must be a well-defined action with no ambiguous aspects. More important, perhaps, from our point of view, and a matter we shall refer to again, is that the solution must be commensurate with the problem. It must not contain within it problems of equal or greater complexity than the original problem. A solution of this type we shall call an *ideal solution*. In practice, we are often content with less-than-ideal solutions.

We cannot pretend that this discussion is logically sound, since it contains concepts that are undefined. Furthermore, some thought will show that such phrases as "all equivalent effects" or "equal or greater complexity" are perhaps

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even more difficult to explain than the ideas with which we started. Since we agreed upon an informal presentation, we shall accept this ambiguity as part of the price we must pay for relative simplicity. Since we know from sad experience that no amount of effort can completely remove logical ambiguities, we should not feel particularly distressed.

4. The solution process. The process or action leading from the problem reduction to the ideal solution we shall call the *solution process*. Thus, a large part of a problem process may be schematically represented as in Fig. 1 below. The connection between this diagram and a flow chart for a computer is not coincidental.

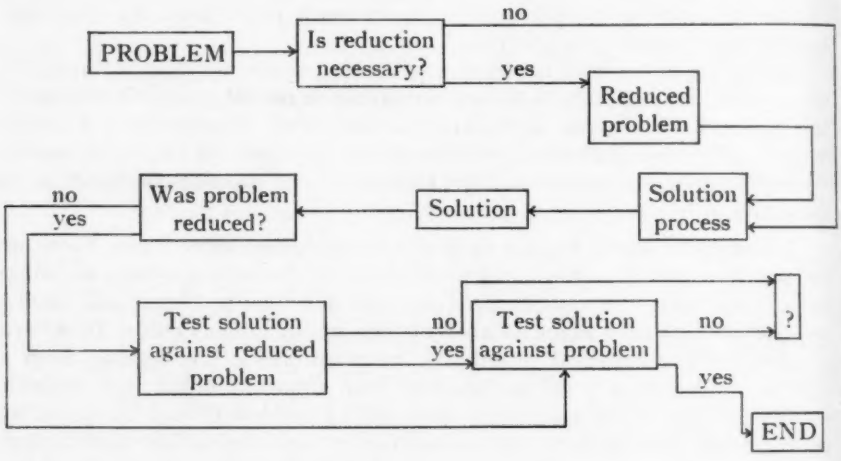

FIG. 1

Problems may not have ideal solutions because the conditions for a solution to be ideal are not satisfied, or because an appropriate solution process cannot be implemented for one of many reasons. This is the usual situation, and is not at all to be considered exceptional.

5. Applications of mathematical techniques to problem solving. Advances of mathematics may be considered to be the development of increasingly more powerful tools to apply to problems that can be transformed into symbolic form. In this connection, we may introduce the notion of a *mathematical model*, as a symbolically representable class problem. Thus, any particular system of linear algebraic equations would fall into the model problem:

$$\sum_{j=1}^{n} a_{ij}x_j = b_i, \qquad i = 1, \cdots, m.$$

Since this model problem has been intensively investigated, it is only necessary to check any particular system against the several possibilities of the model to determine the number of solutions, representations for solutions, exact or approximating procedures to express solutions numerically, and so on.

If a mathematical solution to a symbolic problem exists, it must be an ideal solution in the following sense:

1. The solution must satisfy all conditions of the problem.

2. If not unique, all possible solutions must be expressible in a mathematical inductive sense. For any particular solution, mathematical equivalence must be understood.

3. Any subordinate problems posed in a solution must be solvable by well understood procedures.

It must, however, be emphasized that if a problem has been reduced to mathematical form, the existence of a mathematical solution to the reduced problem does not necessarily imply an ideal solution to the original problem.

6. Types of mathematical difficulty. We consider three classes of difficulty that may exist in a solution process. These are *analytic*, *computational*, and *conceptual*.

1. *Analytic.* At this level, we assume that the problem has been reduced to one of solving a set of equations. In most cases, however, these equations cannot be solved in terms of elementary or tabulated functions. To obtain numerical results from these equations, we must use approximate, or equivalently, iterative techniques such as Rayleigh-Ritz methods, finite difference methods, Monte Carlo methods, the Picard method of successive approximations, power series, and so on. When an analytic problem is transformed into one which permits computation, the limiting operations involved in differentiation, integration, etc., are necessarily replaced by finite operations. In certain cases, the approximating operations may not lead to a solution. But even in the cases where numbers are developed, it remains an analytical problem to determine whether these numbers are ideal solutions in the sense that has been defined. It should be noted that not all analytical problems are capable of solution in terms of existing techniques, and that even reasonable approximations may be beyond us at the moment. In spite of this, however, it is important to emphasize the fact that analytic difficulties are the *simplest* difficulties to overcome.

2. *Computational.* Although it can be argued that this is essentially an analytic difficulty, the revolutionary developments in computation force us to consider it separately. One of the paradoxical aspects of this field of study is the fact that explicit solutions may be of no use at all. A very nice example of this is furnished by the theory of linear equations. Here, Cramer's rule yields the solution as quotients of two determinants. Yet, if the degree of the linear system is large, these determinants cannot be used to grind out numerical results both because of the great number of terms arising from the straightforward evaluation of a

determinant and because of the numerical inaccuracy introduced from "round-offs" at the end of each arithmetical operation. Similarly, in the field of mathematical physics, there are large classes of problems whose solutions can be obtained in the form of infinite series or infinite integrals. In many cases however, the convergence of the expressions is so slow as to make them useless for numerical purposes. This brings to mind the remark of Heaviside: "This series diverges. Now we can compute with it!"

In a number of modern theories, we observe an interesting schizoid effect with one type of analysis used to establish properties such as existence, uniqueness, etc., and another type of analysis used to obtain numbers. Ideally, one would like an approach that can be used simultaneously for both purposes, but there is no reason to suppose that such an approach will always exist. Bigger and faster computing machines permit us to utilize algorithms that smaller machines cannot handle. Furthermore, they help conquer round-off difficulties by retaining more significant figures. Consequently, we can treat more realistic problems in finer detail. There is, however, the very real danger that a number of problems which could profitably be subjected to analysis, and so treated by simpler and more revealing techniques, will instead be routinely shunted to the computing machines. Many examples of this perverse use of these new devices can be cited.

The role of computing machines as a mathematical tool is not that of a panacea for all computational ills.

Although they extend our capabilities in this area, and will continue to do so indefinitely, they will always be of limited applicability. In fact, as will be pointed out subsequently, their prime value to the problem concept may very well be derived from a completely different direction. For this reason it is essential that the role of the computer be analyzed and discussed in great detail.

3. *Conceptual.* It is on the horns of this third dilemma that we are most frequently gored. In all parts of the physical world, and in the even more tortuous terrains of economic, industrial and military applications, the fundamental difficulty lies in formulating a clearly defined mathematical problem. No matter how difficult the equations, or complicated the analysis, we can always make some sort of approximation. What to do when there exists no such precise formulation is a difficulty of a different order of magnitude, which we shall discuss in following sections.

In order to reduce the natural problem to the resolution of equations by means of analytic algorithms, we must construct a qualitative and quantitative model of the physical process, a mathematical model. This is an art rather than a science. It is thus rather remarkable that computing machines can be used to guide the construction of models.

7. On computers. It is important to understand the value and limitations of modern large-scale digital computing equipment, and other types of automatic

computing and data-processing systems that are used in the numerical solution of problems. The digital computers that exist today have many useful attributes, principal among which are:

(a) They perform computations with extreme rapidity.
(b) They have very large memory units.
(c) They are extremely reliable.
(d) They are capable of performing binary choices.
(e) They are capable of automatic sequencing of operations.

The machines that are currently being developed and planned for the future will contain improvement over current equipment, primarily reflected by increased capabilities in items (a), (b), (c), above. If, however, one asks the question, "Will these computers eliminate any over-all computational problem that existed previously?," the answer is categorically "*No!*" Problems can always be formulated that are as intractable time-, accuracy-, and reliability-wise on any proposed computer, as the problems which we are now solving on current equipment were with the use of paper, pencil, and desk calculators. The prime advantage gained by use of the large scale equipment that exists *today* is that problems whose solutions are essential for today's technology are capable *today* of having solutions computed that are compatible with their speed and accuracy requirements. In a sense this is a revolution from an engineering, and data-handling and data-processing point of view. There is little or no gain from a theoretical point of view.

In fact, computers have given rise to a problem of practical significance that previously had only academic interest. We refer to the question of determining whether an arbitrary solution procedure is a proper algorithm. The application to computing machines of Gödel's theorem on logical indecision presents the following anomaly, see [14]. Solution procedures are often attempted where proper algorithms are unknown. Should a result be forthcoming from such a procedure, the algorithm itself is a proper one for at least a partially computable function (but need not be the correct one for the problem in question). If a properly operating computing machine must be stopped because of time considerations alone, one is faced with three possibilities:

(a) The time required for a solution was in excess of that allowed.
(b) The algorithm was proper, but the function values imposed were inadmissible.
(c) The algorithm was improper.

There is no a priori way to determine the applicable possibility.

The mathematician must utilize the computer as an engineer does for maximum efficiency. He may use it in two ways:

1. To obtain computational results which he feels are significant per se, where the computational effort is now reasonable whereas previously by hand and desk computation it was impractical.

2. To perform *mathematical* experiments which formerly were not feasible. We may wish to examine the primality of certain sequences of integers, or the behavior of the solutions of certain systems of nonlinear differential equations, or the effects of certain policies in various types of multistage decision processes.

It is generally recognized that although formal proofs are presented in deductive style, in actuality all great advances have been inductive. Discovery in the field of mathematics, as elsewhere, is a creative process.

It is worth pointing out that the experimental method was standard in the early days of number theory, and even in the early days of algebraic number theory. Much as Gauss divined the asymptotic distribution of primes from tables, and Euler the quadratic reciprocity theorem from examples, Artin formulated the Riemann hypothesis for finite fields as a result of computations based on particular primes. The inductive procedure is now capable of tremendous amplification with the aid of large-bore digital computers. Were a modern computer available to Fermat, he would not have hypothesized that the Mersenne number $2^{2^5}+1$ was a prime. He would have tested the sequence $2^{2^n}+1$ for many values of n before making a statement. On the other hand, with the aid of the SWAC at UCLA, the validity of Fermat's last theorem has been tested to unbelievable limits. It must be confessed that at the present time, very few of the computers are used for these exploratory purposes as contrasted with their use in applying known methods.

8. The principle of balance. To construct a mathematical model, we abstract certain features of the physical situation, make certain assumptions, and then pose an analytic problem whose solution gives us some information concerning the interaction of these features and the assumptions we have made. It may happen, and often does, that a very simple mathematical model gives rise to analytic problems of great intricacy, some of which may be insoluble in terms of present abilities. The simplest examples of this are in the field of number theory.

The question arises as to the physical validity of intricate solutions arising from simple models. As was indicated, it may be very difficult to derive a solution to a natural problem from the solution of a mathematical model of the problem. Concerning the third condition for a solution, however, we can prescribe an important criterion that is often overlooked: We postulate a simple

PRINCIPLE OF BALANCE: *A physically meaningful solution of a mathematical problem arising from a mathematical model of a physical process should never possess a greater degree of complexity than the mathematical model itself.*

Vague as the principle is (all useful principles of wide applicability must be so), we feel that it plays a vital role as a rule-of-thumb guide in the construction of mathematical models. Since the models determine to a great extent the ex-

penditure of research efforts and research funds, it is clear that it is worth devoting careful thought to these matters.

In this context, it must be brutally stated that a good deal of the mathematical work in the domain of applied mathematics in recent years resembles an inverted pyramid. From mathematical models of limited applicability and dubious validity have mushroomed thousands of pages of mathematical symbols, hundreds of papers and technical reports, and dozens of books. This formidable mass of equations constitutes one of the principal roadblocks to the consideration and solution of some of the most significant problems that face us. In each particular case, it must be ascertained whether a detailed analysis of a simplified model is worth as much as a crude analysis of a more realistic model. In the treatment of more realistic models by computational techniques, the computing machines can play a vital constructive role. As an accessory to the crime of turning out more reports with more graphs and more numbers based upon the apex of the inverted pyramid, it can play a quite destructive role.

Before contemplating the construction of bigger and better computers for the solution of specific problems, the foregoing comments should be duly considered.

9. A mathematical classification of problems. The mathematical problems derived from mathematical models can be separated into three categories:

1. *Deterministic problems.* These are problems where the mathematical formulation is in terms of determinate functions, *i.e.*, those having explicit values for specific values of their arguments. This does not imply that the solutions are readily accessible. For example, the problem may resolve into a nonlinear partial differential equation, the theory of which may be insufficiently developed even to guarantee the existence or uniqueness of a solution. On the other hand, the problem may be computationally impossible with existing computing equipment. The difficulties in deterministic problems are, however, always *analytic* or *computational*, never *conceptual*.

2. *Stochastic problems.* A far more important problem that arises practically is one in which all information is not known explicitly. We can consider a class of problems whose underlying structure is well undestood but where the detailed information is stochastic in character. These stochastic variables may have a known distribution function. However, in many cases, they may have unknown distribution functions, or these distributions may be approximate, based on past experiences or other extraneous information.

3. *Adaptive or learning processes.* Problems to which we assign this classification arise when the basic structure of a physical process, and hence of the associated problems, is incompletely known. As we set about determining a solution, we learn more about the process.

Although it may seem impossible at first sight to apply precise mathematical

techniques to problems which are not properly posed, this is actually not the case. The theory of games, sequential analysis, and dynamic programming all afford approaches to problems of this nature that arise from so many different applications.

10. The race against $N!$. In the remaining sections, we wish to illustrate some of the ways in which apparently innocuous problems can give rise to formidable computational and conceptual conundrums. Many of the difficult and significant problems in algebra, analysis, topology, mathematical physics, and the economic, industrial and military spheres are of combinatorial type. Let us discuss a particular problem of this nature; see [1], [2], for many further references.

Suppose that we have N men to be placed in N jobs. Assume that experience has shown that the value of placing the ith man in the jth job is given by a quantity a_{ij}, and that the value of any particular arrangement of these N men in N jobs is given by the sum of the individual values. It is clear that the first man can be placed in any of N different positions. Once this first man has been located, the second man can be placed in any of $(N-1)$ different positions. It follows, pursuing this line of argument, that the total number of possible ways in which N men can be placed in N jobs is $N!$.

If $N=10$, this number is $10!=3{,}628{,}800$, a large number, but not incredibly so. If $N=20$, a number of the same magnitude as 10, the enormity of 20! can perhaps best be understood in the following terms. Given a computing device which can evaluate the worth of any particular arrangement of 20 men in 20 jobs in a *microsecond*, it would require over 500,000 *years* to examine all of the 20! possibilities.

This then is what we mean by the race against $N!$. To treat problems of this type, highly over-simplified versions as they are of problems that arise in scheduling theory, we cannot employ direct enumerative techniques. We must develop analytic techniques and other computational algorithms. Remarkably enough, for the problem posed above, this has been done; references will be found in [1].

11. Approximate policies and mathematical experimentation. As noted above, the "assignment" problem can be resolved in a very short time by a number of elegant techniques. Slight changes, however, in the criterion for efficiency result in problems at present insoluble. It follows that it is of great interest to examine approximate policies of simple nature that can be used in situations of this type. Ideally, we would like to develop techniques that yield good approximations in those cases where we know the answer, to be used for problems without solution at present. It is here that computing machines can play an important role. With their aid, we can carry out mathematical experimentation on a scale undreamt of twenty years ago. The results of these experiments combined with analysis will yield useful approximate policies; see [3] for a discussion of the novel technique of Boldyreff. The concept of mathematical

experimentation is closely related to the study of simulation processes, [4]. These will be discussed below.

12. Detection problems, information patterns. One of the important military uses of computing machines is the SAGE system where these devices are used to keep track of enemy and friendly forces. In the case of enemy forces, it is vital to evaluate the signals of detection devices and particularly to distinguish between actual attackers and false signals arising from decoys. In order to indicate the types of problems that arise from this situation, let us discuss a simple version of the general problem.

"Suppose that we have N coins, all of the same appearance, all of equal weight except one which is heavier, and an equal arm balance. This equal arm balance may be used to weigh one batch of k coins against another, and will furnish the information that one batch is heavier, lighter, or of equal weight. The problem is to determine a weighing procedure which will minimize the maximum number of weighings required to isolate the heavy coin."

This problem, which is a favorite Sunday-supplement puzzle section question, may be easily analyzed for the following reason. If two batches of k coins are compared, two events can occur. Either the two batches balance, or one batch is heavier than the other. In the first case, we have the same problem with $N-2k$ coins, and in the second case, the same problem with k coins. We leave it to the reader to deduce the solution from this observation.

Let us now complicate the problem, by supposing that there are *two* heavy coins. Now if two batches balance, we know only that either batch contains one or no heavy coin. In place of having a simpler problem of the same type, we have a more complicated problem as a result of one observation. Furthermore, each successive test, up to a point, appears to complicate the description of the problem. The solution to this problem is unknown at present.

A problem of military import, of which detection is one part, is that of surveillance. Before we go about putting radar stations on land and sonar stations under water, we must ask ourselves what information we are looking for, how we are going to assess it, and what we are going to do about it. It seems clear that multiplying information sources and increasing sensitivity of detectors need not necessarily increase our effective defense against attack, and could conceivably weaken it, due to the effects of saturation and overload.

The problem of coin-weighing is a simple example of a search problem. Some other particular examples are:

1. Locate the maximum of a function of N variables over a given region of variability.
2. Locate the defective element in a multicomponent device. A particular aspect of this is the problem of writing handbooks for mechanics who must repair electrical or mechanical devices, see [5].

These illustrations are indicative of a fundamental problem in testing, com-

munication, and information theory in general. This is the problem of describing the state of a complex system in some efficient fashion. Both examples are representative of formidable problems in which we must simultaneously devise an information code to describe the state of the system and select a sequence of tests that can utilize this information in a most efficient fashion. A very interesting fact to emphasize is that the amount of information that is required depends heavily upon the policy that is employed. This is one of the basic principles of dynamic programming, [6]. Conversely, the policy that is employed depends equally upon the information that is available.

13. A note on computer design. Any problem involving the design of a computer or a solution by a designed computer must be in the form of a mathematical model. Of particular relevance to the problem of designing computers for specific applications is the question of obtaining "maximum information." If information is assumed to consist merely of an accumulation of data, it is safe to say that increasing the storage capacity of a device increases the amount of information we possess. Hence, in itself, it would seem to be a "good thing" to devote effort to the development of larger memories. However, we must understand how this quantity of information must be used in order to evaluate it. It is very easy to contemplate procedures which decrease in efficiency as the amount of information is increased.

A very simple example is furnished by the academic game of furnishing bibliographies. A few well-chosen papers greatly enhance the value of a paper. However, were a reader to read the fifty or so papers which are occasionally added like a tail to a kite, and then to read the papers cited in the bibliography to these, and so on, it is clear that not only would he never get to any original work, but what is worse, he would never necessarily get to read all the significant papers. It follows that one of the prime functions of an information gathering procedure is that of *rejecting* information.

14. Nonexistence of criteria of efficiency. One of the most baffling aspects of the problems that arise in military, economic, industrial, and sociological domains is the absence of any clear-cut goal. To take a very simple example, consider the problem of radar detection of enemy-manned or unmanned weapons. At first glance, the objective is very simple; we wish to recognize and destroy all enemy objects. In attempting to convert this into engineering terms, we realize very quickly that a *guarantee* of interception and destruction is at the present time beyond our abilities. We must then compromise.

This compromise takes various forms. Without paying any attention to the over-all system, we can concentrate on radar, data-processing, etc., and try to maximize the probability that an enemy object will be detected and that its position will be determined to within a desired accuracy within a prescribed time. It is clear, however, that these specifications cannot be guaranteed if a variety of enemy weapons are employed with appropriate countermeasures. The problem of countering countermeasures brings us into stochastic processes, se-

quential analysis, the theory of games, and the still more recent domain of adaptive processes.

We emphasize that the problem of component design cannot be completely separated from system performance. For surveillance, the specifications for a detection device depend to a great degree upon the type of object it is trying to detect, and the use that will be made of this information. These same difficulties plague mathematical economics.

15. Some nondeterministic examples. The problems treated in the preceding sections have been deterministic. By this we mean that each cause has a unique effect, and that this effect is known in advance. Consequently, the outcome of a sequence of decisions can be calculated before the process under study starts. Since it is obvious that very few processes of any significance will possess this property, it is imperative that we study more realistic situations. The basic problem that faces us in most situations is that of making decisions with incomplete information.

Consider, as illustrative, the problem of directing an aircraft whose aileron control is inoperable. One might ask the question, "How does an airplane fly when its aileron is flapping uncontrolled during its flight?" The equations representing the flight can be explicitly stated up to a functional representative for the displacement of the aileron but this displacement variable now has a stochastic character associated with it.

Problems of this type have a conceptual difficulty associated with them that was not present in the deterministic type problem. Mathematically, we may consider the airplane problem as a differential equation with a stochastic term in it. What is meant by the solution of such an equation? It is quite conceivable that a question of this type has no answer and yet extremely relevant information might be obtained from such a problem. Thus, one might deduce the stability of the solution of a problem of this type for certain distribution functions associated with the stochastic variable while for other types of distributions, one might expect unstable solutions. This class of problems has not been investigated extensively; see [7], [8].

16. The two-armed bandit problem. Stochastic problems divide naturally into two distinct parts. The first is the construction of a mathematical model of ignorance; the second is the application of analytic tools to treat the model. Of these, the most difficult part is that of choosing an appropriate model. It is worth dwelling upon the multiplicity of models, considering the way particular models are overworked in various areas. The impression is formed, justly or not, that the research workers in these areas feel that they are dealing with the "truth," rather than with a particular mathematical model. Let us consider the following levels of uncertainty for a known process structure:

1. The distribution functions for the stochastic effects are known.
2. The distribution functions are determined in functional form, but certain

parameters must be determined on the basis of observation and experiment.

3. The functional forms of the distribution functions are unknown.

Now suppose that we have two slot machines placed before us, each of which will cost us a dollar per play. We are told that these slot machines pay off or not on each play. We are given a limited number of plays to make. The problem is to switch back and forth in the play of the machines in such a way as to maximize the return. On the basis of all that has been said above, it should be clear that the problem as posed is meaningless, since we have not defined what we mean by the terms "return" or "maximize." In order to make it precise, we can proceed in a number of different ways. Let us follow the order of uncertainty given above.

1. The simplest model assumes that the first machine has a probability p_1 of paying a quantity q_1, and that the second machine has a probability p_2 of paying q_2. In addition, we postulate that we shall proceed so as to maximize the expected return. In this case, the solution is simple: we compare p_1q_1 and p_2q_2, and choose whichever machine has the larger product value for the total number of plays.

2. To treat this case, we assume that the p_i and q_i are fixed quantities, but that they are unknown to us. The first question that arises is that of understanding what we mean by maximizing. In order to do this, we must make precise what we mean by the term "unknown." One way of doing that is to assume that the p_i and q_i have been chosen from a given distribution. We then average the return from any particular process over this distribution. This is the usual assumption in statistics. If this assumption is unsatisfactory on the grounds that this distribution is not known, or may not exist, we can introduce the concept of a "*game against nature.*" For any given distribution, we maximize the expected return, and then choose this distribution so as to minimize this maximum expected return; see [9]. Alternatively, for those who feel that this is too pessimistic an outlook, we can merely average over some class of distributions.

3. The third category of problem can be formulated conceptually in the same terms, but the analysis is, in general, quite formidable at the present time [10], [11]. Once again, it is worth pointing out how difficult it is to describe the state of a stochastic process. A complete description of the two-armed bandit problem would require not only a specification of the amount of money obtained to date, but, in addition, the entire history of the process. This may not be a particular barrier as far as computing machine memory is concerned, but it does prevent effective use of any current analytic techniques.

In connection with the detection and tracking of aircraft to avoid air collisions, this accumulation of information could easily overwhelm the facilities of the computers. It follows that in treating these complete problems, we must find ways of rejecting information of little value. We must not think merely in terms of bigger memories and faster access times, since many of the basic problems are not in these directions at all.

17. Simulation processes. In this paper, we are concerned with general classes of mathematical problems and in the extended area of the development of mathematical models from practical considerations.

Any procedure that purports to be in a complete or partial correspondence with a practical procedure, we designate as a simulation procedure. In the most trivial sense, the flight of an aircraft can be used to deduce information about the flight of the aircraft. This would be the identity correspondence. On the other hand, the flight of an aircraft might be simulated by a system of aerodynamic and control equations. The time-dependent solutions would be equivalent to the information obtained by flying the aircraft directly. Although simulation attempts to invoke complete correspondence, it is clear that a complete duplication of a practical situation is impossible. This is because:

(a) It is impossible to consider all the variables involved. In fact, it generally is not known what all the variables are.

(b) It is impossible to represent the exact behavior of the variables of a physical situation either by a nonmathematical simulated situation, or by mathematical function. It is, however, always hoped that the simulation is close enough so that the results of the simulation approximate the true situation sufficiently closely.

Simulation processes are easier, cheaper, and sometimes safer, to construct and analyze than the actual processes. Simulation enables us to observe the effects of certain types of components and policies without worrying unduly about a specific criterion, or the determination of optimal policies. In many situations, simulated models must be used because it is unknown whether or not operational equipment is feasible. The results of the simulation determine the feasibility of constructing practical hardware.

There are many different simulators. These involve particular units of equipment, people, and mathematical representations combined in various proportions. In a full mathematical simulator we are generally faced with a deterministic problem although it is conceivable that the problem may be stochastic in nature. This is particularly true in problems involving random noise. In other types of simulations where mathematical representations are combined with equipment and people, the problem becomes completely stochastic in nature, because the effect of the equipment or of the individuals can only be approximated by stochastic functions. We do not wish to deal extensively with the problem of simulation in this paper. We only point it out as a very important origin of nondeterministic mathematical problems.

18. Adaptive problems. Let us consider the problem of optimizing the play in a chess game. At first glance, this appears to be a finite deterministic problem with a multiplicity of possibilities so extensive that it is beyond the computational facilities that exist today. However, there has never been any indication

that the game is finite in character, or that an underlying move structure exists. Aside from the win-loss requisites, the draw possibilities may be explicitly stated as:

(a) An explicit demonstration that a given board position can have no win-lose possibilities using any allowable line of play. Such a demonstration can be given in the case where two kings exist on the board, and in certain other extremely simple configurations.

(b) A demonstration that the game has entered into a cycle. By a cycle, we mean a sequence of moves leading to a situation identical to that at the beginning of the sequence, followed by the same sequence of moves.

It is entirely possible that the game can continue indefinitely without a win-loss possibility, and without a demonstrable draw situation. As a matter of fact, it has been demonstrated that noncyclic sequences of moves are possible, [12], [13]. The demonstration at any point that a win-loss condition is impossible is just an inductive extension of the over-all analysis of the game itself. Not knowing the structure of the game, we are faced with a classical imponderable question: how to optimize the play in a chess game. It is well known, however, that there are gradations in people's ability to play chess. It is highly unlikely that if the reader entered into a chess match with Bobby Fischer, he would have much of a chance of winning. Why? One is fairly confident that neither Fischer, nor Botvinnik, nor any of the other chess masters have achieved a solution to the optimal play problem that we have stated is currently imponderable. Yet, there is a strong correlation between certain individuals and success at chess.

There are many problems of this type where one can demonstrate that a solution is impossible under stated mathematical conditions, or imponderable and yet a high correlation exists between success in solving such problems and certain individual adaptive behavior. The problem thus resolves itself into establishing a substructure for analysis purposes that may produce some realistic results for problems of this type. We may consider a chess game in the small. Based on a fixed position, one may determine a probabilistic pattern that an opponent will follow. Thus, based upon the move he makes, he will put himself into a demonstratively lost situation or he will put himself in a weakened position, or he will be in a strong position. ("Weak" and "strong" positions are terms that *should* be defined, but *cannot* be in the present state of the art. Nevertheless, they are fairly obvious in meaning to competent chess players.) One might establish a probability pattern for making moves that would vary with the experience of the individuals playing the game and in such a fashion develop locally optimum procedures in a probabilistic sense. There is great interest at the present time in developing computers to play chess. As indicated, the problem is extremely difficult and progress is slow.

This type of reasoning can be applied to many other situations. Consider two opposing commanders facing each other in an essentially unknown tactical

situation. The better commander is one who can apply his forces most effectively against a best estimate of a probability distribution of possible modes of operation that are open to his counterpart. The problem is generally imponderable if a random behavior on the part of each commander is assumed in a war game analysis of such a situation.

19. Conclusion. The principal motivation for this discussion has been the increasing utilization of computer technology to solve problems of all types. We have attempted to analyze those problems that can be formulated mathematically and to show essential differences in problem concept and in problem difficulty.

The role of a computer is twofold:

1. Modern large scale computers have tended to balance practical problem-solution difficulties with the complexities of the problems arising in modern technology. There is every indication that this balance will be difficult to maintain with present computer developments.

2. Computers tremendously extend the range of mathematical testing, experimentation and inductive motivation into ever expanding areas of the mathematical unknown. They certainly tend to expand the cosmopolitan interests of mathematicians.

It must be continually remembered that a computer is not a factotum. When incorrectly used, it can be extremely damaging. Computers do not decrease the need for mathematical analysis, but rather greatly increase this need. They actually extend the use of analysis into the fields of computers and computation, the former area being almost unknown until recently, the latter never having been as intensively investigated as its importance warrants. Finally, it is up to the user of computational equipment to define his needs in terms of his problems. In any case, computers can never eliminate the need for problem-solving through human ingenuity and intelligence.

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RICCATI'S NONLINEAR DIFFERENTIAL EQUATION*

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1. Introduction. There are very few ordinary nonlinear differential equations which can be reduced to the linear form by special transformations. Among these few are the equations of Bernoulli, Jacobi, and Riccati, which have been cited in many books ([1]–[5]). The object of this paper is to extend the "conventional" transform used in solving Riccati's equation.

The so-called generalized Riccati equation is

$$dy/dx + P(x)y + Q(x)y^2 = R(x). \tag{1}$$

The transformation

$$y = z'/(Qz) \tag{2}$$

(where $z = z(x)$ and $z' = dz/dx$), is attributed to Riccati [1]. Since there is another possible transformation for the generalized Riccati equation, the transformation (2) will be called the "conventional" transformation in this paper.

The resultant linear differential equation obtained by using (2) is

$$z'' + [P - (Q'/Q)]z' - QRz = 0, \tag{3}$$

where primes stand for derivatives with respect to x. It is of value to review in a general way what Riccati did. Let

$$y = [u(x)f(x)]/g(x), \tag{4}$$

where $u(x)$, $f(x)$, and $g(x)$ are to be determined. Substitution of y and y' into (1) results in

$$u'fg + uf'g - ufg' + Pufg + Qu^2f^2 = Rg^2. \tag{5}$$

There are two places where derivatives of f and g appear. By making the sum of the third and fifth terms of the left-hand side of (5) vanish; *i.e.*, from $ufg' = Qu^2f^2$, there results

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$$f = g'/(Qu) \tag{6}$$

Therefore, (4) yields

$$y = g'/(Qg). \tag{7}$$

This is the conventional transformation.

Alternatively, by equating the second term of the left- and the right-hand side of (5); *i.e.*, from $uf'g = Rg^2$, there results $g = (uf')/R$. Therefore, (4) yields

$$y = (Rf)/f'. \tag{8}$$

When (8) is substituted into (1), the resultant linear differential equation is

$$f'' - [P + (R'/R)]f' - QRf = 0. \tag{9}$$

Thus, (8) is called the "new" transformation in this paper.

In comparing (3) and (9), it is interesting to note that:

(i) They are second-order variable-coefficient homogeneous linear differential equations.

(ii) The role played by $P(x)$ in (1) is small; this fact is also reflected in (3) and (9).

(iii) $Q=0$ in $y=g'/(gQ)$ gives trouble; yet $R=0$ in $y=(fR)/f'$ gives a trivial solution, provided $(g'/g)\neq 0$ and $(f'/f)\neq 0$, respectively.

The new transformation of (8) does not appear much different from the conventional one; however, their dissimilarities will become evident as higher-order and higher-degree nonlinear differential equations are studied in the following sections.

2. The conventional transformation. The conventional transformation of $y=g'/(Qg)$ is now applied to special forms of higher-order nonlinear differential equations. It would be of interest to find useful transformations which would reduce a typical second-order nonlinear equation such as

$$y'' + A(x)yy' + B(x)y'y^2 + C(x)y + D(x)y^3 = E(x) \tag{10}$$

to the linear form. The method used for extending the conventional transformation can be explained as follows:

Noting that $y=g'/(Qg)$ reduces the Riccati nonlinear (first-order) differential equation, the question raised is: "What type of higher-order nonlinear differential equation does this transformation linearize?" By direct differentiation of (2):

$$y'' = \frac{g'''}{Qg} - \frac{2Q'gg'' + Q''gg'}{Q^2g^2} - \frac{3Qg'g''}{Q^2g^2} + \frac{2Q'^2g'^2}{Q^3g} + \frac{2Q'g'^2}{Q^2g^2} + \frac{2g'^3}{Qg^3}. \tag{11}$$

When (11) is multiplied by g^3, it is observed that in order to eliminate nonlinear terms in g, terms such as $y'y$, and y^3 must be added with appropriate

coefficients. The resultant linear differential equations in g are obtained from these special forms of higher-order nonlinear differential equations in y by the transformation $y=g'/(Qg)$.

The second-order nonlinear differential equation

$$y'' + 3Qyy' + (W/Q)y' + Py + (Q' + W)y^2 + Q^2y^3 = R \tag{12}$$

is similarly obtained, where P, Q, R, and W are arbitrary functions of x. This process can be used to obtain linearizable equations of any order. A special case of (12) has been reported by E. L. Ince ([1], p. 318).

Reducing (12) results in the linear differential equation:

$$g''' - \frac{1}{Q}(2Q' - W)g'' + \frac{1}{Q}\left[PQ - Q'' + Q'\left(2\frac{Q'}{Q} - W\right)\right]g' - QRg = 0. \tag{13}$$

The conventional transformation $y=g'/(Qg)$ will not linearize second-degree first-order nonhomogeneous differential equations (see appendix). In fact, all higher-degree first-order nonhomogeneous differential equations are irreducible by the conventional method.

3. The new transformation. The advantages of the new transformation, *i.e.*, $y=(Rf)/f'$, are:

(i) It handles a certain special group of higher-degree first-order nonhomogeneous differential equations,

(ii) Its resultant linear differential equation does not virtually increase the order by one, while the conventional transformation does.

The first derivative of y is

$$y' = R + \frac{R'f}{f'} - \frac{Rff''}{f'^2}. \tag{14}$$

When y' is raised to the mth power, R^m appears; thus, R^m (a special form of nonhomogeneous term) is required rather than R (a general form of nonhomogeneous term) itself as in (1). The second-degree first-order nonhomogeneous differential equation and its resultant differential equation via $y=(Rf)/f'$ are respectively:

$$(y')^2 - 2Ry' - (R'/R)^2y^2 = -R^2, \tag{15}$$

$$f'' - 2(R'/R)f' = 0. \tag{16}$$

This process of deriving higher-degree first-order nonhomogeneous differential equations can be continued indefinitely. Moreover, this new application of the transformation $y=(Rf)/f'$ also solves nonlinear differential equations of orders higher than the first. For the second-order equation

$$yy'' - 2(y')^2 + 5Ry' - 3R'y = 3R^2, \tag{17}$$

the resultant linear equation is

$$f''' - 2\frac{R'}{R}f'' - \left[\frac{R''}{R} - 2\frac{R'^2}{R^2}\right]f' = 0. \tag{18}$$

This is essentially a second-order linear homogeneous differential equation. Again, this process of deriving any higher-order nonlinear equation of certain special forms continues indefinitely. Of course, as the order and degree of nonlinear differential equations increase, the two transformations, previously mentioned, lose their wide applicability, since the practical engineering and physical systems which utilize these special forms of equations are rare.

4. The extension of Riccati's equation. Two general equations which produce Riccati's equation as special forms are

$$y' + P(x)y + Q(x)y^k = R(x), \tag{19}$$

$$y' + P(x)y + Q(x)y^2 = R(x)y^k. \tag{20}$$

$k=2$ (in (19)) and $k=0$ (in (20)) provide the Riccati equation.

The transformation

$$y = \frac{a(g')^n}{Q^l g^m}, \tag{21}$$

where a, l, m, and n are constants, is applied to (19) as shown below.

$$\begin{aligned} anQ^l g^m (g')^{n-1} g'' - alQ^{l-1}Q' g^m (g')^n - amQ^l g^{m-1}(g')^{n+1} \\ + aPQ^l g^m (g')^n + a^k Q^{l(2-k)+1} g^{m(2-k)} (g')^{nk} = RQ^{2l} g^{2m}. \end{aligned} \tag{22}$$

The sum of the third and fifth terms of the left-hand side of (22) must vanish. Therefore,

$$amQ^l g^{m-1}(g')^{n+1} = a^k Q^{l(2-k)+1} g^{m(2-k)} (g')^{nk}.$$

The results

$$l = m = n = 1/(k-1), \tag{23}$$

$$a = (k-1)^{-1/(k-1)}, \tag{24}$$

$$y = \left[\frac{g'}{Qg(k-1)}\right]^{1/(k-1)}, \tag{25}$$

are obtained by equating exponents of like variables. The remaining terms reduce to

$$g'' + [P(k-1) - (Q'/Q)]g' - R[Qg(g')^{k-2}(k-1)^k]^{1/(k-1)} = 0. \tag{26}$$

Equation (26) can be made linear if $k=2$ for nonzero value of R. If $k \neq 2$ and

$k \neq 1$, the condition $R=0$ is necessary in order to have a resultant linear equation, such as

$$v'' + [P(k-1) - (Q'/Q)]g' = 0. \tag{27}$$

The transformation

$$y = \frac{bR^l g^n}{(g')^m}, \tag{28}$$

where b, l, m, and n are constants, is applied to (20). The result is

$$\begin{aligned} blR^{l-1}R'g^n(g')^m + bnR^l g^{n-1}(g')^{m+1} - mbR^l g^n(g')^{m-1}g^n \\ + bPR^l g^n(g')^m + b^2QR^{2l}g^{2n} = b^k R^{lk+1}g^{nk}(g')^{m(2-k)}. \end{aligned} \tag{29}$$

Equating the second term of the left- and right-hand sides of (29), *i.e.*, from

$$bnR^l g^{n-1}(g')^{m+1} = b^k R^{lk+1}g^{nk}(g')^{m(2-k)},$$

the results

$$= m = n = 1/(1-k), \tag{30}$$

$b=(1-k)^{1/(k-1)}$, and

$$y = \left[\frac{Rg(1-k)}{g'}\right]^{1/(1-k)}, \tag{31}$$

are derived in a manner similar to that used in obtaining (23), (24), and (25).

The remaining terms are arranged as

$$g'' - [P(1-k) + (R'/R)]g' - Q[R(1-k)^{2-k}g(g')^{-k}]^{1/(1-k)} = 0. \tag{32}$$

Equation (32) becomes a linear equation if $k=0$ for $Q \neq 0$.

Except for the special cases of $k=2$ and $R \neq 0$ in (19) and of cases $k=0$ and $Q \neq 0$ in (20), the two transformations of (25) and (31) are identical.

The Bernoulli equation is the case of $Q=0$ in (20) as shown below:

$$y' + P(x)y = R(x)y^k. \tag{33}$$

A transformation discovered by Leibnitz [1] is $Z=y^{1-k}$ where $k \neq 1$, $k \neq 0$, and $Z=Z(x)$. Equation (33) then becomes

$$Z' + P(1-k)Z = R(1-k). \tag{34}$$

The new transformation for reducing Bernoulli's equation (33) is (31), which reduces (33) to

$$g'' + [P(1-k) + (R'/R)]g' = 0. \tag{35}$$

Equation (34) is a nonhomogeneous equation, while (35) is essentially a first-order first-degree homogeneous differential equation. For easier methods of evaluating integrals, (35) has the advantage over (34).

Appendix. For simplicity, let a resultant linear equation be $g''=0$. Then

$$g' = a, \qquad g = a(x+b), \tag{A-1}$$

where a and b are constants. The result of substituting these values in (7) is $y=1/[Q(x+b)]$, and, from (A−1),

$$y' = -\frac{Q'}{Q^2(x+b)} - \frac{1}{Q(x+b)^2}. \tag{A-2}$$

Then, from (A−2),

$$(y')^2 = \frac{Q'^2}{Q^4(x+b)^2} + \frac{2Q'}{Q^3(x+b)^3} + \frac{1}{Q^2(x+b)^4}$$

and hence $(y')^2-(Q'/Q)^2y^2-2Q'y^3-Q^2y^4=0$, which is a homogeneous equation.

For the transformation $y=(Rg)/g'$, application of (A−1) yields

$$y = R(x+b), \qquad y' = R'(x+b)+R. \tag{A-3}$$

Then, from (A-3), $(y')^2-(R'/R)^2y^2-2R'y=R^2$, which is a nonhomogeneous equation.

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A THEOREM ON SELF-OVERLAPPING SEQUENCES

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1. Introduction. A finite sequence of n symbols will be said to be *self-overlapping* if the initial subsequence consisting of the first m terms is identical with the subsequence consisting of the final m terms for some m such that $0<m<n$. The sequences obtainable from a given sequence by the process of removing some initial subsequence and adding it on to the end of the sequence (*i.e.*, cycling) will be defined as *cyclically equivalent.* It will be shown that all sequences cyclically equivalent to a given sequence are self-overlapping if, and only if, the sequence is periodic in the sense that it is composed of an integral number (exceeding one) of identical subsequences.

For the sake of convenience let us refer to a subsequence consisting of the first m terms as a *prefix*, and similarly let the last m terms of the sequence be defined as a *suffix*.

Consider the sequence 011001110. The prefix consisting of the first term, a zero, is the same as the suffix consisting of the last term. Hence we can say that the sequence overlaps itself in that term, and this fact can be emphasized by writing the sequence as shown below:

011001110

011001110.

Now if the sequence is transformed by cycling the first term around to the end to obtain 110011100, another prefix, 1100, will be found to be identical to a suffix, and again the sequence is self overlapping, as shown below:

110011100

110011100.

The sequence 111001100 is cyclically equivalent to those considered above, but it is *not* self-overlapping.

THEOREM. *Every member of the set of sequences that are cyclically equivalent to a given sequence is self-overlapping if, and only if, the given sequence is made up of a chain of at least two identical subsequences.*

(In other words, if a sequence is not periodic in the sense defined above, then there exists a cyclically equivalent sequence that is not self-overlapping.)

Proof.

1. Suppose a sequence consists of a chain of two or more identical subsequences. Then the prefix comprised of the first subsequence is identical with the suffix comprised of the last subsequence and hence the sequence is self-overlapping. Since all cyclically equivalent forms of any periodic sequence are also periodic, it follows that the sufficiency portion of the theorem is true.

The remainder of the proof deals with the question of necessity. Given any sequence we shall show how it can be manipulated into a cyclically equivalent form that is self-overlapping only if it is periodic. The next eight steps consist of a partitioning process.

2. Choose an arbitrary element x, and cycle the sequence so that it starts with an x but does *not* end with an x. This can always be done unless the sequence contains only x's, in which case it is periodic.

3. Partition the sequence into E_0-elements, each beginning with a block of consecutive x's and ending just before the next x block. Let the first element of the sequence begin the first E_0-element. This uniquely defines the E_0's once step 2 has been accomplished.

4. From the set of E_0's choose those with x-blocks of *maximum* length. Select one of these with *minimum total* length and call it E_0^*. Also label as E_0^*-

elements all E_0's identical to the chosen E_0^*.

At this point let us illustrate steps 2–4, with an example. Start with the sequence,

$$abbbdcbaacbbbaacbbbdcbbacbbbc$$

Let $x=b$ and cycle the sequence to obtain

$$bbbdcbaacbbbaacbbbdcbbacbbbca.$$

The E_0's are then enclosed in parentheses to yield

$$(bbbdc)\ (baac)\ (bbbaac)\ (bbbdc)\ (bbac)\ (bbbca).$$

In accordance with step 4, E_0^* could be chosen as either $(bbbdc)$ or $(bbbca)$. If the former choice is made, then, indicating the E_0^*'s by underscoring, we obtain

$$\underline{(bbbdc)}\ (baac)\ (bbbaac)\ \underline{(bbbdc)}\ (bbac)\ (bbbca).$$

Resuming the proof:

5. If the sequence now consists entirely of E_0^*'s then the partitioning process is over. Otherwise we continue by first cycling the sequence so that it starts with an E_0^*.

6. Partition the sequence into E_1-elements by drawing partition lines just prior to the beginning of each E_0^*. Then the first E_1 begins with the first member of the sequence (an E_0^*) and contains all E_0's (if any) up to the next E_0^*, which begins the next E_1, etc. This is a unique process once steps 2–5 have been accomplished.

7. Choose one of the E_1-elements of minimum length (in terms of the original elements of the sequence) and call it, and all other E_1's identical to it, E_1^*-elements.

8. The above process can be continued as follows. Assume that the sequence has been partitioned into E_{i-1} and E_{i-1}^*-elements. If there are only E_{i-1}^*-elements then we stop the partitioning. Otherwise we cycle the sequence to start with an E_{i-1}^*. Then we form E_i-elements, each starting with an E_{i-1}^* and containing no other E_{i-1}^* (just as the E_1's were formed from the E_0^*'s and E_0's). One of the minimum length E_i-elements is then designated as E_i^* along with all other E_i's identical with it.

9. The induction process just described must terminate at some finite n, at which point the sequence will consist of a single E_k^* or of several E_k^*'s. This follows from the finiteness of the sequence and the fact that, for all i, the number of E_i^*-elements is less than the number of E_{i-1}^*-elements.

Now we shall discuss the overlap situation in terms of the partitions described above. Note that, during the partitioning process, no E_i or E_i^*-element (for any i) was ever split.

Suppose that a self-overlap is possible for the final form of the sequence.

Henceforth, the terms prefix and suffix will be reserved for the specific prefix and suffix that make the sequence self-overlapping. The subsequence common to both will be termed the overlap, with length $m<n$ (the length of the sequence).

10. Since the sequence terminates with the end of an E_i (or E_i^*) it follows that if any part of some E_i (or E_i^*)-element ($i \leqq n$) appears in the suffix, then the end of that element also appears there.

11. If an E_0^* begins in the prefix, then it must also be concluded there. The proof follows.

(a) Beginning with the first x of the E_0^*, its x's must all be matched in the suffix. Since no E_0 can have more x's than an E_0^* (see step 4 of this proof) it must follow that this block of x's is the beginning of an E_0 or E_0^* in the suffix.

(b) According to step 10 the element in the suffix must be complete there, and hence the overlap extends at least as far as the end of that element.

(c) But the E_0^* in the prefix starts concurrently with the E_0 or E_0^* in the suffix and cannot exceed it in length since, by definition of E_0^* (step 4), no E_0 with an x-block as long as the x-block of an E_0^* can be shorter than an E_0^*. Therefore there must be room in the prefix for the E_0^* to be complete.

12. Now we shall generalize the above result to show that, for all i, if an E_i^* starts in the prefix then it must be complete there. The proof is inductive. Assume that the statement is true for E_{j-1}^*. (We have shown it to be true for E_0^* in step 11.) Assume now that an E_j^*-element begins in the prefix.

(a) The E_j^* starts with an E_{j-1}^* and hence the E_{j-1}^* must be complete in the prefix (and therefore complete in the suffix as well).

(b) The E_{j-1}^* in the suffix is the start of an E_j or E_j^*. According to 10 this latter element must be complete in the suffix.

(c) But the E_j^* in the prefix starts concurrently with the E_j or E_j^* in the suffix and is no longer (by definition of the E_j^*—step 8). Hence there must be enough room in the prefix for the E_j^* to be complete there.

13. Now consider the sequence in the form of a chain of one or more identical E_j^*-elements (9). The initial E_k^* of the sequence starts in the prefix and hence, according to 12, must also end there. If this single E_k^* constituted the entire sequence, then the overlap would consist of the entire sequence, thus contradicting our assumption that $m<n$. Hence there must be at least two E_k^*'s, and the sequence is periodic. This concludes the proof.

Example. Let us continue the example presented after step 4. The underscored subsequences below are E_0^*'s, and the others are E_0's:

$$\underline{(bbbdc)}\ (baac)\ (bbbaac)\ \underline{(bbbdc)}\ (bbac)\ (bbbca).$$

The next step is to form E_1-elements, of which there are two (corresponding to the two E_0^*'s) shown in parentheses:

$$(bbbdcbaacbbbaac)\ (bbbdcbbacbbbca).$$

The second one is the shorter and it is therefore selected as E_1^* and written first

by cycling the sequence to obtain

$$\underline{(bbbdcbbacbbbca)}\ (bbbdcbaacbbbaac).$$

In the next step we obtain a single E_2-element, consisting of the entire sequence, so that the whole sequence is an E_2^* as follows:

$$\underline{bbbdcbbacbbbcabbbdcbaacbbbaac}$$

The reader may verify the fact that no overlap is possible.

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BASIC PROPERTIES OF PANDIAGONAL MAGIC SQUARES

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1. The following investigation originated from a suggestion by Chater and Chater [1] that all pandiagonal magic squares of even order are of a simple complementary pattern, and from a note by D. F. Lawden [2] showing that for such squares the determinant vanished. The question arose as to the validity of this conjecture for even squares and the possibility of a corresponding property for odd squares. If a_{rs} is the number standing in the rth row and the sth column of a square of order $2n$ the simple complementary property is that $a_{rs}+a_{r+n,s+n}=$constant. It is shown that this simple property, whilst holding for all squares of order 4, does not hold in general but can be replaced by rather more complicated properties having a complementary character. From the basic line-invariants of the type $\sum_{r=1}^{n} a_{rs}=$constant, for the square of order n, other invariants are deduced which prove to be of interest as regards the construction of magic squares and the determination of the number of independent elements in such a square. The patterns corresponding to these invariants are intriguing. Further an octagonal matrix emerges which would seem to be fundamental.

In a pandiagonal magic square the sum of the elements in any row, in any column, in any diagonal (straight or broken) add up to a constant S, the number for the square. It follows immediately that if in any pandiagonal magic square the first column is shifted to a position immediately following the last column, the new square is also pandiagonal magic. Similarly if several columns or rows are so shifted, the square remains pandiagonal magic. Thus if the square is

repeated indefinitely to either side and the resulting infinite strip is repeated indefinitely above and below, the whole plane is filled "parquet-wise." A square frame of side n placed on the plane and adjusted so as not to cut the smaller cells, will enclose a pandiagonal magic square. We may speak of these extra n^2-1 squares so formed as "variants" of the original square. Again, it follows immediately from the definition, that if any two pandiagonal magic squares of the same order are added together as matrices (*i.e.*, corresponding elements are added) the resulting square is pandiagonal magic. Hence we have,

THEOREM 1. *The matrix sum of any pandiagonal magic square and any constant multiples of its variants produces a square which is also pandiagonal magic.*

We have from here on, subtracted S/n from every element in the square so that the sum of the elements in any row, column or diagonal will be zero, and we call such modified zero-sum squares, for short, "z-squares." Since, added as matrices, the sum of any two pandiagonal squares of the same order is itself a pandiagonal square, it follows that a z-square is also a pandiagonal magic square from which those squares with constant S can be immediately obtained.

It is convenient, following Chater and Chater [1], to introduce operators R, C such that

$$Ra_{r,s} = a_{r-1,s}, \qquad Ca_{r,s} = a_{r,s+1},$$

so that R converts any row into the row immediately below that row, and by the parquet property, it converts the last row into the first one; hence $R^n=1$ and similarly $C^n=1$.

Before dealing with the general z-square of order n it is of interest to investigate those of orders 4, 5 and 6.

2. The z-square of order 4. Since the sum of the elements in any column, in any row, and in any diagonal (straight or broken) is zero, we have, operating on any element a_{rs}, the following line-invariants,

(1) column: $$(1 + R + R^2 + R^3)a_{rs} = 0,$$

(2) row: $$(1 + C + C^2 + C^3)a_{rs} = 0,$$

(3) diagonal: $$(1 + RC + R^2C^2 + R^3C^3)a_{rs} = 0,$$

(4) $$(R^3 + R^2C + RC^2 + C^3)a_{rs} = 0.$$

Subtracting (3)–(1) gives

$$R(C - 1)[1 + R(C + 1) + R^2(C^2 + C + 1)]a_{rs} = 0.$$

Hence $[1+R(C+1)+R^2(C^2+C+1)]a_{r+1,s}=Q_{r+1}$ (say), where Q_{r+1} is a quantity depending on the row $r+1$ but independent of the column s. Clearly,

$$\sum_{s=1}^{4} Q_{r+1} = [1 + R(C + 1) + R^2(C^2 + C + 1)] \sum_{s=1}^{4} a_{r+1,s} = 0$$

and therefore $Q_{r+1}=0$; hence $[1+R(C+1)+R^2(C^2+C+1)]a_{r+1,r}=0$. Since this is true for all values of r, s it is convenient to suppress the operand a_{rs} when using these operational formulae so that we may write

$$1 + R(C + 1) + R^2(C^2 + C + 1) = 0. \tag{5}$$

The elements involved in (5) form a triangle and we call this invariant the triangle-invariant. (See diagram (a).)

1	•	•	•
1	1	•	•
1	1	1	•
•	•	•	•

diagram (a)

1	1	•	•
1	1	•	•
•	•	•	•
•	•	•	•

diagram (b)

By symmetry, we may in (5) interchange the operators R, C so that

$$1 + C(R + 1) + C^2(R^2 + R + 1) = 0 \tag{6}$$

and subtracting (5)–(6) gives $(R - C)[1 + R + C + RC] = 0$, so that $[1+R+C+RC]a_{rs}=Q$, where Q is a quantity which is constant along a diagonal, upwards from left to right, and hence $\sum Q$ vanishes when taken along this diagonal, and hence $Q=0$. Hence we have established the "square-invariant" diagram (b),

$$1 + R + C + RC = 0. \tag{7}$$

From the diagrams (a), (b) it can be seen that a simpler invariant is obtained by subtracting the square-invariant from the triangle-invariant. In fact taking (5)–R(7), we obtain

$$1 + R(C + 1) + R^2(C^2 + C + 1) - R(1 + R + C + RC) = 0,$$

$$1 + R^2C^2 = 0. \tag{8}$$

This is the simple complementary property $a_{r,s}+a_{r+2,s+2}=0$ and must therefore hold generally for all z-squares of order 4.

The existence of the invariants (2), (7), (8), namely, the row, square, and complementary invariants, enables us to construct the general z-square of order 4 as economically as possible, that is, with the use of the minimum number of specified elements and therefore not involving resultant simultaneous equations from which redundant elements must be eliminated. To sketch the process briefly: Set down the elements a_{11}, a_{12}, a_{13}; then (2) determines a_{14}. Set down the element a_{21}; then (7) determines a_{22} and in turn a_{23} and then by (2) or (7), a_{24}. The first two rows being complete, the remainder of the z-square follows immediately from the complementary property (8).

If we call the independent elements a, b, c, d, the general z-square of order 4 then becomes,

a	b	c	$-a-b-c$
d	$-a-b-d$	$a-c+d$	$b+c-d$
$-c$	$a+b+c$	$-a$	$-b$
$-a+c-d$	$-b-c+d$	$-d$	$a+b+d$

It is of interest here to notice that if we put $a=1$, $b=c=d=0$, we obtain the pattern of diagram (c) in which, by extending parquet-wise to the next two rows, gives the octagonal pattern of diagram (d). By putting in turn $b=1$, $c=1$,

1	·	·	−1
·	−1	1	·
·	1	−1	·
−1	·	·	1

diagram (c)

·	1	−1	·
−1	·	·	1
1	·	·	−1
·	−1	1	·

diagram (d)

$d=1$, and the rest of the independent elements zero, we obtain patterns which are reducible to the simple octagon or to sums of octagons. Notice that the octagon as shown in diagram (d) is itself a z-square and if bordered with the appropriate number of rows and columns of zeros, still remains a z-square of any order, and hence when added to a z-square the result is still a z-square. Hence in view of (8) we have,

THEOREM 2. *All z-squares of order* 4 *have the simple complementary property* $a_{r+2,s+2}=a_{r,s}$, *and for all orders of z-squares greater than* 4, *there exist z-squares which do not possess this property.*

3. The z-square of order 5. Exactly as in Section 2 we have the two line-invariants,

$$1+R+R^2+R^3+R^4=0, \tag{9}$$

$$1+RC+R^2C^2+R^3C^3+R^4C^4=0. \tag{10}$$

1	·	·	·	·
1	1	·	·	·
1	1	1	·	·
1	1	1	1	·
·	·	·	·	·

diagram (e)

1	1	1	·	·
1	2	1	·	·
1	1	1	·	·
·	·	·	·	·
·	·	·	·	·

diagram (f)

·	·	·	−1	·
·	1	·	·	·
·	·	1	·	·
−1	·	·	·	·
·	·	·	·	·

diagram (g)

·	·	·	·	·
·	1	1	·	·
·	1	1	·	·
·	·	·	·	·
·	·	·	·	1

diagram (h)

Subtracting (10)–(9) and justifying the removal of the factor $R(C-1)$ we have immediately the corresponding triangle-invariant shown in diagram (e),

$$1 + R(1 + C) + R^2(1 + C + C^2) + R^3(1 + C + C^2 + C^3) = 0. \tag{11}$$

Again as in Section 2, interchanging R, C, subtracting, and justifying the removal of the factor $R-C$, we obtain the square-invariant of diagram (f),

$$(1 + R + R^2)(1 + C + C^2) + RC = 0. \tag{12}$$

The pattern of these squares is beginning to emerge and shows that the weighting increases towards the center of the square.

If we invert the triangle (e) and subtract from the square (f), (we can most easily work from the diagrams), we have

$$\begin{aligned}(1 + R + R^2)(1 + C + C^2) + RC \\ - [1 + C + C^2 + C^3 + R(1 + C + C^2) + R^2(1 + C) + R^3] = 0,\end{aligned}$$

$$- R^3 - C^3 + RC + R^2C^2 = 0, \tag{13}$$

as given in diagram (g). The negative terms of (13) may be removed by adding in a diagonal-invariant of the type (4), *viz.*, $C^3+RC^2+R^2C+R^3+R^4C^4=0$. This yields the invariant

$$RC(1 + R)(1 + C) + R^4C^4 = 0, \tag{14}$$

which is shown in diagram (h). It will be realized later that this is the counterpart for a square of order 5, of the simple complementary property already found for squares of order 4.

There is a further invariant which is important and may be derived as follows: The triangle-invariant (11) remains valid if we replace C by $1/C$ as this merely represents a reflection in a vertical line, and gives

$$1 + R(1 + C^{-1}) + R^2(1 + C^{-1} + C^{-2}) + R^3(1 + C^{-1} + C^{-2} + C^{-3}) = 0.$$

Subtracting this from (11) then gives

$$R(C - C^{-1})[1 + R(1 + C + C^{-1}) + R^2(2 + C + C^{-1} + C^2 + C^{-2})] = 0,$$

and since the line-invariant $C^{-2}+C^{-1}+1+C+C^2=0$, this becomes

$$R(C - C^{-1})[1 + R(C + 1 + C^{-1}) + R^2] = 0.$$

Hence if $[1+R(C+1+C^{-1})+R^2]a_{rs}=Q_{rs}$, then $Q_{r,s-1}=Q_{r,s+1}=Q_{r,s+3}=\cdots=Q$ (say). Now since $Q_{r,s+5}=Q_{r,s}$, it follows that these terms are equal, each being $=Q$, and further summing throughout a row, $\sum Q=0$, hence $Q=0$. So that we have an invariant

$$1 + R(C + 1 + C^{-1}) + R^2 = 0 \tag{15}$$

in which the terms form a diamond as shown in diagram (i). It must be emphasized that the division by the factor $C-C^{-1}$ is only permissible in a square of odd order and can be made to depend on the fact that $1+C^n$ has a factor $1+C$ for all odd values of n. This would seem to be the fundamental difference be-

tween squares of odd and even orders, *viz.*, the existence or nonexistence of a diamond invariant of the type (15).

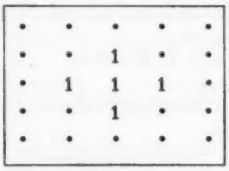

diagram (i)

diagram (j)

We are now in a position to construct the general z-square of order 5 as economically as possible: Lay down the first four elements in each of the first two rows, namely $a_{11}, a_{12}, a_{13}, a_{14}, a_{21}, a_{22}, a_{23}, a_{24}$; the line-invariant (9) then gives a_{15} and a_{25}. Since the first two rows are now complete, repeated application of the diamond-invariant gives all the elements of the third row and further repetitions give also the fourth and fifth rows. Alternatively these can be found using the invariant (14) to give the fourth row and the fifth row then follows by using the column-invariants of the type (9). Thus it follows that the general z-square of order 5 can contain, at most, eight independent elements.

The general z-square of order 5 may be written as,

a	b	c	d	$-a-b-c-d$
e	f	g	h	$-e-f-g-h$
$g+h-a$	$-b-e-f-g$	$-c-f-g-h$	$-d+e+f$	$a+b+c+d+f+g$
$-c-d-g-h$	$a+b+c+e+f+g$	$b+c+d+f+g+h$	$-a-b-e-f$	$-b-c-f-g$
$c+d-e$	$-a-b-c-f$	$-b-c-d-g$	$a+b-h$	$b+c+e+f+g+h$

Again it is of interest to note that putting all but one, in turn, of the independent elements equal to zero, the corresponding patterns which result are each easily expressible in terms of the octagonal matrix in diagram (d).

Other symmetrical invariant patterns can easily be produced, for example, that of diagram (j) which is found by subtracting the diamond (15) from the square of diagram (f). These are of use as a check.

4. The z-square of order 6. The fundamental processes having been established we give only the main invariant patterns arising for a z-square of order 6 in the diagrams $(k), \cdots, (p)$.

In the process of constructing the general z-square of order 6, three rows of five elements must be laid down, together with one element of the fourth row, so that the invariant of diagram (o) or of diagram (p) can be applied. The fifth

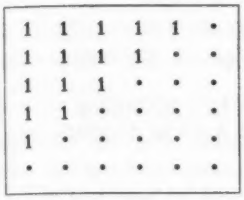

diagram (k)

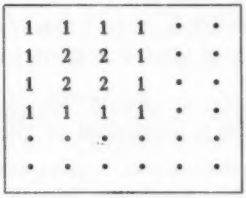

diagram (l)

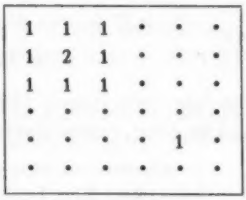

diagram (m)

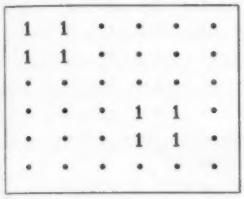

diagram (n)

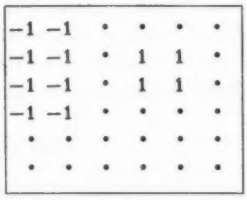

diagram (o)

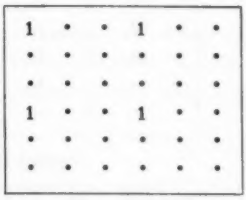

diagram (p)

row then follows by diagram (m) and the square is then completed using the column-invariant. Hence 16 independent elements are required.

Again it is found that by putting all independent elements save one, equal to zero, the various patterns obtained are each expressible in terms of the octagonal matrix. This gives rise to the conjecture that z-squares of any order can be built up entirely from numerical multiples of the octagon.

It may be mentioned that diagram (o) is derived from diagram (n) by subtracting the column-invariant from each of the first and second columns. Diagram (p) is derived from diagram (l) by multiplication by $1+R^3$ and subtracting the column-invariant from each of the first four columns. This invariant is peculiar to a z-square of order 6 and is particularly useful in its construction.

5. General considerations. z-squares of order n. It is now fairly evident that we can separate those invariants which arise generally from those which only arise in special cases, *e.g.*, the "five-spot" of diagram (j), the open-square of diagram (p) and the complementary pair (7). It follows easily from the earlier methods that for the general z-square of order n, by subtracting $\psi(RC)-\psi(R)$, where

$$\psi(R) \equiv 1 + R + \cdots + R^{n-1}, \tag{16}$$

and taking out a factor $R(C-1)$, we obtain the triangle-invariant with side $n-1$ given by

$$1 + R(1 + C) + R^2(1 + C + C^2) + \cdots + R^{n-2}(1 + C + \cdots + C^{n-2}). \tag{17}$$

Interchanging R, C, subtracting, and dividing by $R-C$ we obtain the square-invariant of side $n-2$,

$$(18)\qquad \begin{aligned}&(1+R+\cdots+R^{n-3})(1+C+\cdots+C^{n-3})\\&\quad+RC(1+R+\cdots+R^{n-5})(1+C+\cdots+C^{n-5})+\cdots,\end{aligned}$$

the last term being $(RC)^{(n-3)/2}$ or $(RC)^{(n-4)/2}(1+R)(1+C)$ according as n is odd or even, respectively. This is illustrated for the case $n=7$ in diagram (q).

1	1	1	1	1	•	•
1	2	2	2	1	•	•
1	2	3	2	1	•	•
1	2	2	2	1	•	•
1	1	1	1	1	•	•
•	•	•	•	•	•	•
•	•	•	•	•	•	•

diagram (q)

1	1	1	1	•	•	•
1	2	2	1	•	•	•
1	2	2	1	•	•	•
1	1	1	1	•	•	•
•	•	•	•	•	•	•
•	•	•	•	•	1	•
•	•	•	•	•	•	•

diagram (r)

Subtracting the triangle-invariant from (18) and adding in the diagonal-invariant gives the invariant with two complementary squares of sides 1 and $n-3$ shown in diagram (r).

Alternatively, subtracting the row-invariant from each of the first $n-2$ rows of the square-invariant (q), (giving diagram (s)), and adding the column-invariant to the last two columns produces the invariant consisting of two complementary squares of sides 2 and $n-4$, as shown in diagram (t). These constitute the first two members of a sequence of "complementary squares" invariants.

•	•	•	•	•	−1	−1
•	1	1	1	•	−1	−1
•	1	2	1	•	−1	−1
•	1	1	1	•	−1	−1
•	•	•	•	•	−1	−1
•	•	•	•	•	•	•
•	•	•	•	•	•	•

diagram (s)

•	1	1	1	•	•	•
•	1	2	1	•	•	•
•	1	1	1	•	•	•
•	•	•	•	•	•	•
•	•	•	•	•	1	1
•	•	•	•	•	1	1
•	•	•	•	•	•	•

diagram (t)

•	•	1	•	•	•	•
•	1	1	1	•	•	•
1	1	2	1	1	•	•
•	1	1	1	•	•	•
•	•	1	•	•	•	•
•	•	•	•	•	•	•
•	•	•	•	•	•	•

diagram (u)

Finally, in the triangle-invariant, by replacing C by $1/C$ and subtracting, we have generally for any z-square of odd order a diamond-invariant of the type in diagram (u).

The construction of the general z-square of order n now becomes automatic using the methods already set out. First lay down arbitrarily a matrix of $n-3$ rows and $n-1$ columns. The rows are completed by the row-invariants.

For the case in which n is odd the diamond-invariant will complete the $(n-2)$th row and then an invariant of type (r) completes the $(n-1)$th row. The square is then completed by the use of column-invariants.

For the case in which n is even an additional arbitrary member is necessary in the $(n-2)$th row; this enables invariants of the type (s) to be used to complete the $(n-2)$th row. The square is then completed in the same way as in the previous case.

This gives the following theorem:

THEOREM 3. *The number of independent elements in a general z-square of order n cannot exceed $(n-1)(n-3)$ if n is odd, or $(n-2)^2$, i.e., $(n-1)(n-3)+1$ if n is even.*

It is conjectured that the general z-square will actually require these number of independent constants. (The general pandiagonal magic square would of course require one extra arbitrary element.)

6. The octagonal matrix. It is convenient to represent the matrix $a_{r,s}$ by a polynomial $F(R, C)$ given by

$$F(R, C) = \sum_{r=1}^{n} \sum_{s=1}^{n} a_{r,s} R^{r-1} C^{s-1}. \tag{19}$$

If, in this way, the octagonal matrix is represented by $\phi(R, C)$ then from diagram (d),

$$\begin{aligned} \phi(R, C) &= -R(1-R)(1-C^2) + C(1-C)(1-R^2) \\ &= (1-R)(1-C)[C(1+R+R^2) - R(1+C+C^2)] \\ &= -(1-R)(1-C)(R-C)(1-RC). \end{aligned} \tag{20}$$

The symmetry and simplicity of this representation indicates the fundamental nature of the octagonal matrix. It has been mentioned in Section 2 that wherever the octagonal matrix is placed on the "parquet" it forms, when bordered by a suitable number of rows and columns of zeros, a z-square of order n for all values of n. Combining this fact with Theorem 1 we have,

THEOREM 4. *The matrix represented by $F(R, C)$ such that*

$$F(R, C) = f(R, C)\phi(R, C) \tag{21}$$

is a z-square for all polynomials $f(R, C)$.

This theorem gives immediately an easy method for the construction of z-squares and hence also of pandiagonal magic squares.

It is possible to establish a converse of Theorem 4 in the following way:

If the polynomial $F(R, C)$ represents a z-square, then for each value of s, $\sum_{r=1}^{n} a_{r,s}=0$, so that $F(1, C)=0$ identically in C, and hence $F(R, C)$ is divisible by $1-R$. Let the quotient be $f(R, C)$. Now if we define $f_1(R, C)$ by the equation,

$$f_1(R, C) = f(R, C)[1 - n^{-1}\psi(R)], \tag{22}$$

where $\psi(R)$ is given by (16), we have immediately,

$$(1 - R)f_1(R, C) = (1 - R)f(R, C) - n^{-1}f(R, C)(1 - R)\psi(R) = F(R, C), \tag{23}$$

since $(1-R)\psi(R)=0$, so that $f_1(R, C)$ is a particular quotient of $F(R, C)/(1-R)$. Further, since $\psi(1)=n$, $1-n^{-1}\psi(R)$ has a factor $1-R$; in fact

$$1 - n^{-1}\psi(R) = (1 - R)\psi_1(R),$$

where

$$n\psi_1(R) = n + (n - 1)R + (n - 2)R^2 + \cdots + R^{n-2}. \tag{24}$$

Thus $F(R, C)/(1-R)=f(R, C)(1-R)\psi_1(R)=\psi_1(R)F(R, C)$, and hence by Theorem 4, there exists a particular quotient $F(R, C)/(1-R)$ which represents a z-square.

Similarly by interchanging R, C the result also holds for division by $1-C$.

Again from (19) if we put $\alpha=RC$ we have

$$F(R, C) = \sum_{r=1}^{n}\sum_{s=1}^{n} a_{r+s,s}R^{r-1}(RC)^{s-1} = F^*(\alpha, C). \tag{25}$$

In precisely the same manner $F^*(1, C)=0$ identically in C and the process shows that $F^*(\alpha, C)/(1-\alpha)$ also represents a z-square. Replacing C by $1/C$ shows that the same result holds for division by $R-C$.

Combining the results for successive divisions by $1-R$, $1-C$, $R-C$, $1-RC$, and noting the form of ϕ in (20) we have;

THEOREM 5. *If $F(R, C)$ represents a z-square then there exists a particular quotient $F(R, C)/\phi(R, C)=f(R, C)$ such that $f(R, C)$ also represents a z-square.*

This implies that the field of z-squares is, in this sense, closed under the operator ϕ.

References

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UNIFORMLY CONTINUOUS SETS IN METRIC SPACES

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It is well known that every function which is continuous on a compact set is uniformly continuous on the set. The converse, however, is not true; that is, there exist sets which are not compact but which possess the property that every function which is continuous on the set is uniformly continuous on the set. N. Levine has characterized linear sets having the above property provided the image space is the set of real numbers.*

It is the purpose of this paper to further investigate sets each of which has the property that every continuous function from the set into M_1 is uniformly continuous on the set, where (M_1, d_1) is a given metric space.

DEFINITION. *Let (M, d) and (M_1, d_1) be metric spaces. Then a subset E of M is said to be U.C. (M_1) if and only if every function from E into M_1 which is continuous on E is uniformly continuous on E.*

DEFINITION. *Let (M, d) be a metric space and E a subset of M. Then E is said to U.I. (uniformly isolated) if and only if there exists a real number $\eta > 0$ such that $d(p, q) \geqq \eta$ for every $p \neq q$ in E.*

THEOREM 1. *Let (M, d) be a metric space and E a subset of M. If E is U.C. (M_1), where (M_1, d_1) is a metric space containing a simple arc axb, then (E, d) is a complete metric space.*†

Proof. Suppose (E, d) is not complete. Then there is a Cauchy sequence $\{p_i\}_{i=1}^{\infty}$ of distinct points of E such that $\bigcup_{i=1}^{\infty} p_i$ has no limit point in E. Define $A = \bigcup_{i=1}^{\infty} p_{2i-1}$, $B = \bigcup_{i=1}^{\infty} p_{2i}$. Then A and B are disjoint closed subsets of E. Since (E, d) is normal, there exists a continuous function f from E into R_1 such that $0 \leqq f(p) \leqq 1$ for every $p \in E$, $f(p) = 0$ if $p \in A$, $f(p) = 1$ if $p \in B$.§ There is a homeomorphism h of the closed unit interval $[0, 1] \subset R_1$ onto the arc axb of M_1 such that $h(0) = a$, $h(1) = b$. It follows that the restriction $g = h/f(E)$ of h to $f(E) \subset [0, 1]$ is continuous on $F(E)$. The function $k = gf$ defined by $k(p) = g[f(p)]$ for every $p \in E$ is clearly a continuous function from E into M_1 which is not uniformly continuous on E. This gives a contradiction and the theorem is proved.

COROLLARY. *Let (M, d) be a metric space and E a subset of M. If E is U.C. (M_1), where (M_1, d_1) is a metric space containing a simple arc axb, then E is closed.*

* N. Levine, Uniformly Continuous Linear Sets, this MONTHLY, vol. 62, 1955, pp. 579–580.

† The authors are indebted to the referee for this theorem.

§ D. W. Hall and G. L. Spencer, Elementary Topology, New York, 1955, pp. 110–121.

THEOREM 2. *Let (M, d) and (M_1, d_1) be metric spaces and E a subset of M. Then a sufficient condition for E to be U.C. (M_1) is that $E=E_1\cup E_2$, where E_1 is compact and E_2 is U.I.*

Proof. The proof is similar to that of Theorem 2 in the previously mentioned article by Levine.

DEFINITION. *Let (M, d) be a metric space and E a subset of M. Then E is U.C. $(R_1, +)$ if and only if every nonnegative real-valued function which is continuous on E is uniformly continuous on E.*

THEOREM 3. *Let (M, d) be a metric space and E a subset of M. Then a necessary and sufficient condition that E be U.C. $(R_1, +)$ is that E be U.C. (R_1).*

Proof. The sufficient condition is obvious. To prove the necessary condition, assume $E\neq\varnothing$ (the theorem is trivial if $E=\varnothing$), let E be U.C. $(R_1, +)$ and let f be any continuous real-valued function on E. Define functions f_1 and f_2 from E into R_1 as follows: for every $p\in E$, let $f_1(p)=f(p)$ if $f(p)\geqq 0$ and $f_1(p)=0$ if $f(p)<0$; and let $f_2(p)=0$ if $f(p)\geqq 0$ and $f_2(p)=f(p)$ if $f(p)<0$. Then for every $p\in E$, $f(p)=f_1(p)+f_2(p)$. Each of f_1 and f_2 is continuous on E. Moreover, each of f_1 and $-f_2$ is nonnegative and continuous on E and hence uniformly continuous on E. It follows that f_2 is uniformly continuous on E and $f=f_1+f_2$ is uniformly continuous on E, and the theorem is proved.

DEFINITION. *A metric space (M, d) is said to be a W-B space if and only if every closed and bounded subset of M is compact.*

Remark. The property of being a W-B space is not a topological property. (R_1, d_{R_1}), d_{R_1} being the standard metric, is a W-B space but (I^0, d_{R_1}) where I^0 is the open interval $0<x<1$ is not a W-B space although I^0 is homeomorphic to R_1.

THEOREM 4. *If E is a subset of a W-B space (M, d) and E is U.C. (R_1), then $E=E_1\cup E_2$, where E_1 is compact and E_2 is U.I.*

Proof. E is U.C. (R_1) implies that E is U.C. $(R_1, +)$. The theorem is trivial if $E=\varnothing$. Therefore assume $E\neq\varnothing$. Let p_* be a fixed point in E and consider the set $\{N(p_*, r)^- \mid r\in R_1\}$ where $N(p_*, r)^-$ is the closure of $N(p_*, r)$. There exists a point $r_*\in R_1$ such that $E-N(p_*, r_*)^-$ is U.I. For if not, then $E-N(p_*, r)^-$ is U.I. for no $r\in R_1$ implies that there exists a sequence $\{p_i\}_{i=1}^{\infty}$ of distinct points in E such that:

(i) $\lim r_i=\infty$ where $r_i=d(p_i, p_*)$; $i=1, 2, \cdots$,

(ii) $d(p_{2i-1}, p_{2i})<1/i$; $i=1, 2, \cdots$,

(iii) $r_1\leqq r_2<r_3\leqq r_4<\cdots<r_{2i-1}\leqq r_{2i}<r_{2i+1}\leqq\cdots$.

Furthermore, there is no loss in generality in assuming

(iv) $r_{2i}+2\leqq r_{2i+2}$ for all $i=1, 2, \cdots$.

It is easily verified that:

(i) $d(p_{2i}, p_{2j})\geqq 2$ if $i\neq j$,

(ii) $p_{2j-1} \notin N(p_{2i}, s_i)^-$, $i = 1, 2, \cdots$ for each $j = 1, 2, \cdots$, where $s_i = d(p_{2i-1}, p_{2i})/2$ for each $i = 1, 2, \cdots$.

(iii) $N(p_{2j}, s_j)^- \cap N(p_{2i}, s_i)^- = \varnothing$ if $i \neq j$.

A nonnegative function f from E into R_1 which is continuous on E but is not uniformly continuous on E can now be constructed contrary to E being $U.C.$ $(R_1, +)$. Define f from E into R_1 as follows: for every $p \in E$, let $f(p) = 1$ if $p \notin N(p_{2i}, s_i)^-$ for all $i = 1, 2, \cdots$ and $f(p) = d(p_{2i}, p)/s_i$ if $p \in N(p_{2i}, s_i)^-$ for some i. Then $0 \leqq f(p) \leqq 1$ for every $p \in E$. In particular, $f(p) = 0$ if $p = p_{2i}$ for some i and $f(p) = 1$ if $p = p_{2i-1}$ for some i.

It follows that $E - N(p_*, r_*)^-$ is $U.I.$ for some r_*. Define $E_2 = E - N(p_*, r_*)^-$. Since $E = (E \cap N(p_*, r_*)^-) \cup (E - N(p_*, r_*)^-)$, let $E_1 = E \cap N(p_*, r_*)^-$. E_1 is the intersection of two closed sets and hence is closed. Also, $E_1 \subset N(p_*, r_*)^-$, which implies that E_1 is bounded. It follows then that E_1 is compact and the theorem is proved.

THEOREM 5. *Let (M, d) be a W-B space and E a subset of M. If E is $U.C.$ (M_1) where (M_1, d_1) is a metric space containing a simple arc axb, then $E = E_1 \cup E_2$, where E_1 is compact and E_2 is $U.I.$*

Proof. Assume $E \neq \varnothing$ (since otherwise the theorem is trivial), and let p_* be a fixed point in E. There exists a $r_* \in R_1$ such that $E - N(p_*, f_*)^-$ is $U.I.$ For if not, then proceeding as in the proof of Theorem 4 we define the same continuous function f from E into $[0, 1] \subset R_1$. There is a homeomorphism h of the closed unit interval $[0, 1] \subset R_1$ onto the arc axb of M_1 such that $h(0) = a$, $h(1) = b$. It follows that the restriction $g = h/f(E)$ of h to $f(E) \subset [0, 1]$ is continuous on $F(E)$ in R_1. The function $k = gf$ defined by $k(p) = g[f(p)]$ for every $p \in E$ is a continuous function from E into M_1 which is not uniformly continuous on E. This is contrary to E being $U.C.$ (M_1). Hence $E - N(p_*, r_*)^-$ is $U.I.$ Define $E_2 = E - N(p_*, r_*)^-$ and $E_1 = E \cap N(p_*, r_*)^-$. E_1 is closed and bounded and hence compact; and the conclusion follows.

It is obvious that the following theorem is true.

THEOREM 6. *Let (M, d) be a metric space and E a subset of M. If (M_1, d_1) is a metric space containing a simple arc axb, then a necessary condition that E be $U.C.$ (M_1) is that $E = E_1 \cup E_2$, where E_1 is closed and bounded and E_2 is $U.I.$*

THEOREM 7. *Let (M, d) be a W-B space, E a subset of M and F a subset of E. If E is $U.C.$ (M_1), where (M_1, d_1) is a metric space containing a simple arc axb, and if F is closed in (M, d), then F is $U.C.$ (M_1).*

Proof. This is an immediate consequence of Theorems 2 and 5.

COROLLARY. *Let (M, d) be a W-B space and $\theta = \{E\}$ a collection of subsets of M. If for every $E \in \theta$, E is $U.C.$ (M_1), where (M_1, d_1) is a metric space containing a simple arc axb, then $\bigcap_{E \in \theta} E$ is $U.C.$ (M_1).*

LEMMA. *Let (M, d) be a metric space and E a nonempty subset of M. If E is*

dense-in-itself and if $E=E_1\cup E_2$, *where* E_1 *is closed,* E_2 *is U.I., and* $E_1\cap E_2=\varnothing$, *then* $E_2=\varnothing$.

Proof. Assume $E_2\neq\varnothing$ and let $p\in E_2$. Then $p\in E\subset E'$. There exists a real $\delta_1>0$ such that $N(p,\delta_1)\cap E_1=(N(p,\delta_1)\cap E_1)-\{p\}=\varnothing$; for if not, p is a limit point of E_1 and hence $p\in E_1$ which is contrary to $E_1\cap E_2=\varnothing$. $E_2\neq\varnothing$ and is *U.I.* implies that there exists a real $\delta_2>0$ such that $d(p,q)\geqq\delta_2$ for every point $q\in E_2$ for which $p\neq q$. Let $\delta=\min(\delta_1,\delta_2)$. It follows that $N(p,\delta)\cap E=\{p\}$ and hence $(N(p,\delta)\cap E)-\{p\}=\varnothing$ which is contrary to $p\in E'$.

THEOREM 8. *Let* (M,d) *be a metric space and E a subset of M. If E is dense-in-itself, then E is U.C.* (R_1) *if and only if E is compact.*

Proof. If E is compact, then every real-valued function f which is continuous on E is uniformly continuous on E and hence E is *U.C.* (R_1).

Conversely, let E be *U.C.* (R_1) and show that E is compact. Assume $E\neq\varnothing$ since otherwise the theorem is trivial. By Theorem 6, $E=E_1\cup E_2$ where E_1 is closed and bounded and E_2 is *U.I.* Furthermore, there is no loss in generality in assuming that $E_1\cap E_2=\varnothing$. It then follows from the lemma that $E_2=\varnothing$.

It remains to show that $E=E_1$ is compact. (E,d) is a complete metric space. Therefore, E is compact if and only if E is totally bounded. If E is not totally bounded, then there exists a real number $\epsilon_1>0$ such that $E\not\subset\bigcup_{i=1}^{n}N(p_i,\epsilon_1)$ for every finite number n of points in E. Hence starting from an arbitrary point $p_1\in E$, an infinite sequence $\{p_i\}_{i=1}^{n}$ of distinct points in E can be constructed such that $d(p_i,p_j)\geqq\epsilon_1$ if $i\neq j$.

Now $N(p_i,\epsilon_1/4)^-\cap N(p_j,\epsilon_1/4)^-=\varnothing$ if $i\neq j$ lest $d(p_i,p_j)\leqq\epsilon_1/2<\epsilon_1$. Define a second sequence $\{p_i^*\}_{i=1}^{n}$ of distinct points in E as follows: $p_i^*\in N(p_i,\epsilon_1/4)$ such that (i) $d(p_i,p_i^*)<\epsilon_1/8i$ and (ii) $p_i\neq p_i^*$. It follows that $p_i^*\neq p_j^*$ if $i\neq j$, and $p_i^*\neq p_j$ for all $i,j=1,2,\cdots,n,\cdots$. Let $s_i=d(p_i,p_i^*)>0$ and consider $\{N(p_i,s_i)^-\,|\,i\in P\}$. If $i\neq j$, then $N(p_i,s_i)^-\cap N(p_j,s_j)^-=\varnothing$ and $d^*[N(p_i,s_i),N(p_j,s_j)]>0$.

Define a function f from E into R_1 as follows: for every point $p\in E$, let $f(p)=1$ if $p\notin N(p_i,s_i)^-$ for all $i=1,2,\cdots$ and $f(p)=d(p_i,p)/s_i$ if $p\in N(p_i,s_i)^-$ for some i. Then $0\leqq f(p)\leqq 1$ for every $p\in E$. f is continuous on E but is not uniformly continuous on E contrary to E being *U.C.* (R_1). It follows that E is totally bounded and hence compact, and the theorem is proved.

THEOREM 9. *Let* (M,d) *be a metric space, E a subset of M, and* (M_1,d_1) *a metric space containing a simple arc axb. If E is dense-in-itself, then E is U.C.* (M_1) *if and only if E is compact.*

Proof. The sufficient condition is obvious. The proof of the necessary condition is easily patterned after the proofs of Theorems 6 and 8.

SKEW MATRICES AS SQUARE ROOTS*

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1. In the algebra of matrices it has been observed that the skew hermitian matrices exhibit a number of properties similar to those of pure imaginary numbers in the algebra of complex numbers. The present note exhibits one such property. Skew hermitian matrices are characterized as the normal square roots of negative definite, or semidefinite, hermitian matrices H. They constitute a set of generators of all like-ranked square roots of such hermitian matrices in the sense that every such square root is similar to a skew hermitian square root.

Similar results hold for the corresponding real case. Thus, with a certain additional hypothesis on H, the last two statements above remain valid if the phrases "skew hermitian," "square root" and "hermitian" are replaced respectively be "real skew," "real square root" and "real symmetric."

2. THEOREM 1. *Every skew hermitian matrix is a (normal) square root of a negative definite, or semidefinite, hermitian matrix. Conversely, every negative definite, or semidefinite, hermitian matrix possesses matrix square roots, the normal matrices among which are skew hermitian.*

Proof. Let S be skew hermitian (hence normal). Then $S^* = -S$, where S^* is the conjugate transpose of S. Hence, $S^2 = -S^*S$. Since $(S^*S)^* = S^*S$, S^2 is hermitian. Further, S^*S, being the so-called Gram matrix of S [1], is known to be positive definite or semidefinite. Hence S^2 is a negative definite, or semidefinite, hermitian matrix.

For the converse, let H be hermitian, negative definite or semidefinite. We first show that normal square roots of H do exist. Since H is hermitian $\exists$ a unitary matrix V such that V^*HV is a diagonal matrix, $\text{diag}(\lambda_1, \cdots, \lambda_n)$, and all the λ_i are real numbers. Because H is negative definite or semidefinite, the λ_i are all nonpositive. The diagonal matrix $D = (\sqrt{\lambda_1}, \cdots, \sqrt{\lambda_n})$ has pure imaginaries or zeros on the diagonal, hence is skew hermitian, therefore normal, and $D^2 = V^*HV$. Hence VDV^* is skew hermitian, and $(VDV^*)^2 = VD^2V^* = H$.

Now let N be any normal square root of H, *i.e.*, $NN^* = N^*N$, $N^2 = H$. Since N is normal, $\exists$ a unitary matrix $U \ni U^*NU$ is a diagonal matrix, $\text{diag}(\mu_1, \cdots, \mu_n)$. Hence $\text{diag}(\mu_1^2, \cdots, \mu_n^2) = (U^*NU)^2 = U^*N^2U = U^*HU = \text{diag}(\lambda_1, \cdots, \lambda_n)$. The λ_i, being eigenvalues of H, are all nonpositive real numbers. Hence in $U^*NU = \text{diag}(\mu_1, \cdots, \mu_n)$, each μ_i must be pure imaginary or zero, and consequently $\text{diag}(\mu_1, \cdots, \mu_n)$ is skew hermitian. Therefore $N = U\ \text{diag}(\mu_1, \cdots, \mu_n)U^*$ is skew hermitian. The corresponding real companion of Theorem 1 is

THEOREM 2. *Every real skew matrix is a (real, normal) square root of a negative definite, or semidefinite, real symmetric matrix, whose nonzero eigenvalues have even multiplicities. Conversely, every negative definite, or semidefinite, real symmetric matrix, whose nonzero eigenvalues have even multiplicities, possesses real square roots, the normal ones of which are real skew.*

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Proof. Let S be a real skew (hence normal) matrix. Since S^2 is a real hermitian matrix, we conclude exactly as before that S^2 is real, symmetric (*i.e.*, real hermitian), negative definite or semidefinite. We further observe that since the eigenvalues of a real skew matrix consist of zeros (possibly), and pairs of nonzero conjugate pure imaginary numbers, the nonzero eigenvalues of S^2, being the squares of the nonzero eigenvalues of S, consist of pairs of equal negative real numbers. The distinct nonzero eigenvalues of S^2 must therefore have even multiplicities.

For the converse, let H be an nth order real symmetric (*i.e.*, real, hermitian) negative definite, or semidefinite, matrix, whose nonzero eigenvalues have even multiplicities. As before, we first show that such an H admits real normal square roots. Since H is real, symmetric, $\exists$ a real orthogonal matrix P, such that

$$P^T H P = \operatorname{diag}(-\lambda_1, -\lambda_1, -\lambda_2 - \lambda_2, \cdots, -\lambda_k, -\lambda_k, 0 \cdots 0) = D,$$

and the λ_i are all real and positive since H is real, symmetric, and negative definite or semidefinite. Then the nth order matrix F consisting of the diagonal 2×2 blocks

$$\begin{pmatrix} 0 & \sqrt{\lambda_1} \\ -\sqrt{\lambda_1} & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & \sqrt{\lambda_2} \\ -\sqrt{\lambda_2} & 0 \end{pmatrix}, \cdots, \begin{pmatrix} 0 & \sqrt{\lambda_k} \\ -\sqrt{\lambda_k} & 0 \end{pmatrix},$$

and zeros otherwise, is real, skew, and satisfies $F^2 = D$. Hence (PFP^T) is real, skew, and satisfies $(PFP^T)^2 = H$.

Now let S be any normal real square root of H. Exactly as in the proof of Theorem 1 we conclude that S is skew hermitian, and since S is real, it is real skew.

3. If B is any square root of a square matrix C, and if Q is a nonsingular matrix commuting with C, then $Q^{-1}BQ$ is also a square root of C. Thus from the normal (skew hermitian) square roots of H discussed in Section 2, can be obtained other square roots of H, not necessarily normal. That not all square roots of H are so obtainable is shown by the matrix

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

which is a square root of the negative semidefinite hermitian matrix

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Since

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

is not similar to a diagonal matrix, it is not similar to a normal square root of

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

We note that the ranks of the square root matrix and the argument matrix are not the same in the example. However, if S is a normal (therefore skew hermitian) matrix square root of a negative definite or semidefinite matrix H, then the rank of S must be the same as that of H, since $S^2 = -S^*S$, and from known properties of the Gram matrix, the rank of S^*S, *i.e.*, of H, is the same as that of S. Hence any square root of H which is similar to S must also have the same rank as S and H.

Further, the normal square roots of H provide a set from which can be generated by similarity transformations all those square roots of H which have the same rank as H. Specifically,

THEOREM 3. *Let H be hermitian negative definite, or semidefinite, of rank r. Then every square root of rank r of H is similar to a skew hermitian square root of H.*

The proof of Theorem 3 is facilitated by appealing to the following

LEMMA. *If A is a matrix of rank r which is similar to a diagonal matrix, then any kth root of A is similar to a diagonal matrix.*

The lemma is a direct consequence of application of a method suggested by Weitzenböck [2] of finding all kth roots of a given matrix, or, less directly, by application of the method of Franklin [3] for solving polynomial equations $p(X) = A$.

Now let B be a square root of H of the same rank as H. By the lemma there is a nonsingular matrix P such that $P^{-1}BP = \text{diag}(\mu_1, \cdots, \mu_n)$ and $(P^{-1}BP)^2 = P^{-1}B^2P = P^{-1}HP = \text{diag}(\mu_1^2, \cdots, \mu_n^2)$. The μ_i^2 are the eigenvalues of H, and hence are nonpositive real numbers. Hence each μ_i is zero or pure imaginary and $P^{-1}BP$ is skew. Now let U be a unitary matrix such that $U^*HU = \text{diag}(\mu_1^2, \cdots, \mu_n^2)$. Then $S = U\,\text{diag}(\mu_1, \cdots, \mu_n)U^*$ is a skew hermitian square root of H, and $B = PU^*SUP^{-1} = (UP^{-1})^{-1}S(UP^{-1})$. Thus B is similar to a skew hermitian square root of H.

One notes that $(UP^{-1})^{-1}H(UP^{-1}) = H$, and hence UP^{-1} commutes with H. Thus all like-ranked square roots of H with the same rank as H are obtained by taking similarity transforms of the skew hermitian square roots of H, using transforming matrices which commute with H.

In case H is nonsingular, then any square root of H must be nonsingular *i.e.*, must have the same rank as H. Hence,

COROLLARY 3.1. *If H is a nonsingular negative definite, or semidefinite, hermitian matrix, then every square root of H is similar to a skew hermitian square root of H.*

In case H, hermitian negative definite, or semidefinite, has distinct eigen-

values, it is easily seen from the Weitzenböck analysis that the diagonalized form U^*HU of H, with U unitary, has only diagonal, hence skew hermitian, square roots, whence all the square roots of H are given by UDU^*, D skew hermitian. Hence,

COROLLARY 3.2. *If H is hermitian negative definite, or semidefinite, with distinct eigenvalues, then every square root of H is skew hermitian.*

The real analogue of Theorem 3 is:

THEOREM 4. *Let H be a real symmetric negative definite, or semidefinite, matrix of rank r, whose nonzero eigenvalues have even multiplicities. Then every real square root of rank r of H is similar, over the real field, to a real skew square root of H.*

Proof. Let R be a real square root of H of the same rank as H. By the lemma, as in the proof of Theorem 3, R is similar to a diagonal matrix $D = P^{-1}RP$, whose nonzero diagonal entries are conjugate pure imaginaries and which can be taken to be a direct sum of blocks of the form

$$C = \begin{pmatrix} i\lambda & 0 \\ 0 & -i\lambda \end{pmatrix},$$

λ real, and a zero matrix; then $P^{-1}HP = D^2$ is composed of corresponding diagonal blocks

$$\begin{pmatrix} -\lambda^2 & 0 \\ 0 & -\lambda^2 \end{pmatrix}$$

and a zero block. Now the unitary matrix

$$U = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}$$

transforms C into

$$U^*CU = \begin{pmatrix} 0 & \lambda \\ -\lambda & 0 \end{pmatrix}.$$

Hence a unitary matrix V which is a direct sum of such U and an identity block, transforms D into V^*DV, a direct sum of blocks of the form

$$\begin{pmatrix} 0 & \lambda \\ -\lambda & 0 \end{pmatrix}$$

and a zero matrix, and leaves $P^{-1}HP$ unaltered, $V^*P^{-1}HPV = P^{-1}HP$. Thus $K = V^*DV$ is a real skew square root of

$$P^{-1}HP = \operatorname{diag}(-\lambda_1^2, -\lambda_1^2, \cdots, -\lambda_k^2, -\lambda_k^2, 0 \cdots, 0).$$

Let W be a real orthogonal matrix such that $W^THW = P^{-1}HP$. Then $WKW^T = S$

is a real skew square root of H, and $R = PDP^{-1} = PVKV^*P^{-1} = (PVW^T)(WKW^T)(WV^*P^{-1}) = (WV^{-1}P^{-1})^{-1}S(WV^{-1}P^{-1})$. Hence R is similar to the real skew square root S of H, but the transforming matrix $WV^{-1}P^{-1}$ may be complex. However, since the real matrices R and S, viewed as matrices over the complex field, have the same invariant factors, they have, as matrices over the real field, likewise all invariant factors in common and hence are similar over that field. Hence $\exists$ a real matrix Q such that $R = Q^{-1}SQ$.

As in Section 3, there follows

COROLLARY 4.1. *If H is a nonsingular negative definite, or semidefinite, real symmetric matrix, then every real square root of H is similar over the real field to a real, skew square root of H.*

Since a real symmetric negative definite, or semidefinite, matrix which admits real skew square roots must have even multiplicities on its nonzero eigenvalues, a real analogue of Corollary 3.2 would have to read something like: "If H is a real symmetric negative definite, or semidefinite, matrix, whose eigenvalues all have multiplicity two, then every square root of H is real, skew." Such an analogue is not valid as the following example shows. Let

$$H = \begin{pmatrix} -\alpha^2 & 0 \\ 0 & -\alpha^2 \end{pmatrix},$$

α real and $\neq 0$. Then any real matrix M similar to

$$S = \begin{pmatrix} 0 & \alpha \\ -\alpha & 0 \end{pmatrix}$$

is a real square root of H, and not all such M are real, skew.

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CORRECTIONS

H. P. Mulholland and C. A. B. Smith, An inequality arising in genetical theory, this MONTHLY, vol. 66, 1959, pp. 673–683. The right-hand side of (1), page 674 should be $(v^Tuv)^n$.

H. L. Turrittin, Linear differential or difference equations with constant coefficients, this MONTHLY, vol. 66, 1959, pp. 869–875. Equations (6) should be $\overline{X} = PY_f(t)\overline{K}_1$ and $\overline{X}(t) = PY_f(t)\overline{K}_2(t)$.

MATHEMATICAL NOTES

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A GENERALIZATION OF CEVA'S THEOREM

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Ceva's theorem in elementary geometry deals with a triangle $A_1A_2A_3$ and a point P in general position in its plane. The lines A_1P, A_2P, A_3P, intersect the sides A_2A_3, A_3A_1, A_1A_2, respectively, in points I_1, I_2, I_3. The theorem states that

$$\frac{A_2I_1}{A_3I_1}\cdot\frac{A_3I_2}{A_1I_2}\cdot\frac{A_1I_3}{A_2I_3}=(-1)^3.$$

We propose to generalize this result by considering a plane polygon $A_1(x_1, y_1)$, $\cdots$, $A_n(x_n, y_n)$, and $\frac{1}{2}d(d+3)-1$ points (P) in general position in its plane.

A preliminary example, with $n=5$, $d=2$, will help to clarify what follows. Each vertex A_r of a pentagon $A_1A_2A_3A_4A_5$ determines with four fixed points $P_1P_2P_3P_4$ a unique conic Q_r. Name the six points in which A_1A_2 intersects Q_3, Q_4, and Q_5, P_{12}^1 $(i=1, \cdots, 6)$; name the six points in which A_2A_3 intersects Q_4, Q_5, and Q_1, P_{23}^i; define similarly P_{34}^i, P_{45}^i, P_{51}^i. What is to be proved is that the product

$$(A_1P_{12}^1\cdots A_1P_{12}^6)(A_2P_{23}^1\cdots A_2P_{23}^6)\cdots(A_5P_{51}^1\cdots A_5P_{51}^6)$$

is equal to $(-1)^5$ times the similar product taken in the opposite direction, *i.e.*, to

$$(-1)^5(A_1P_{51}^1\cdots A_1P_{51}^6)(A_5P_{45}^1\cdots A_5P_{45}^6)\cdots(A_2P_{12}^1\cdots A_2P_{12}^6).$$

The theorem can now be stated in general terms.

THEOREM. *Let $Q_r(x, y)=0$ be the unique curve of degree d determined by the points (P) and the vertex* A_r of a plane polygon $A_1, \cdots, A_n$. Let $A^s_{r(t,t+1)}$ denote the product of all the signed lengths of the segments joining A_s to the d points of intersection of $Q_r=0$ with the line A_tA_{t+1}. Then*

$$\prod_{r=1}^{n}\prod_{s=r+1}^{r+n-2}\frac{A^s_{r(s,s+1)}}{A^{s+1}_{r(s,s+1)}}=(-1)^n.$$

Proof. The proof that follows uses a result of Newton's which we recall.† Let the two points $B(b_1, \cdots, b_m)$, $C(c_1, \cdots, c_m)$ determine a line in euclidean

* We shall identify A_h and A_k provided $h\equiv k \pmod n$.

† See G. Salmon, Higher Plane Curves, 3rd ed., Dublin 1879, p. 108.

m-space which intersects $H(x_1, \cdots, x_m)=0$, a hypersurface of order g. Then

$$P_b/P_c = H(b_1, \cdots, b_m)/H(c_1, \cdots, c_m), \tag{1}$$

where P_b is the product of the directed distances from B to the g points of intersection and P_c, the corresponding product for C. When this last formula is applied to A_s, A_{s+1}, and the intersection of the line passing through them with the curve $Q_r=0$, we obtain

$$\frac{A^{s}_{r(s,s+1)}}{A^{s+1}_{r(s,s+1)}} = \frac{Q_r(x_s, y_s)}{Q_r(x_{s+1}, y_{s+1})}$$

and hence

$$\prod_{s=r+1}^{r+n-2} \frac{A^{s}_{r(s,s+1)}}{A^{s+1}_{r(s,s+1)}} = \frac{Q_r(x_{r+1}, y_{r+1})}{Q_r(x_{r+n-1}, y_{r+n-1})} = \frac{Q_r(x_{r+1}, y_{r+1})}{Q_r(x_{r-1}, y_{r-1})} = J_r.$$

To show that $\prod_1^n J_r=(-1)^n$, write Q_r as $\sum_{i+j\leq d} {}_ra_{ij}x^iy^j$. Since all the Q_r together form a linear pencil, the points $\bar{x}_r$ in $\frac{1}{2}d(d+3)$-space, whose homogeneous coordinates are ${}_ra_{ij}$, lie on a straight line. Furthermore, $\bar{x}_r$ is the unique point of intersection of this line (L) with the hyperplane

$$W_r = \sum_{i+j\leq d} (x_r)^i(y_r)^j X_{ij} = 0.$$

It is also clear that $W_p(\bar{x}_r)=Q_r(x_p, y_p)$ for any r, p. Applying (1) to the points $\bar{x}_{r+1}$, $\bar{x}_{r-1}$ and the intersection $\bar{x}_r$ of the line (L) through them with the hyperplane $W_r=0$, we obtain

$$\prod_1^n \frac{\bar{x}_{r-1}\bar{x}_r}{\bar{x}_{r+1}\bar{x}_r} = \prod_1^n \frac{W_r(\bar{x}_{r-1})}{W_r(\bar{x}_{r+1})} = \prod_1^n \frac{Q_{r-1}(x_r, y_r)}{Q_{r+1}(x_r, y_r)} = \prod_1^n J_r.$$

Hence $\prod_1^n J_r$ turns out to be the product of the n distances $\bar{x}_r\bar{x}_{r+1}$ taken in one direction divided by the same product taken in the opposite direction, and so, must equal $(-1)^n$.

The same method may readily be extended to prove the theorem for $A_1, \cdots, A_n$ in three or more dimensions so long as the number of points (P) (which would now be in general position in the space determined by the A's) is such that the points (P) will determine with each A_r a unique hypersurface.

We can also deduce two geometrical interpretations of the cross ratio (c) of any linear pencil of four d-ic hypersurfaces. First it is clear that $c=(\bar{x}_1\bar{x}_2\bar{x}_3\bar{x}_4)$, where the $\bar{x}_i$ are related to the four hypersurfaces Q_i as above. Secondly, choose A_1 arbitrarily on Q_1 and A_2 on Q_2. Then

$$\frac{A^{1}_{3(1,2)}}{A^{2}_{3(1,2)}} \div \frac{A^{1}_{4(1,2)}}{A^{2}_{4(1,2)}} = \frac{Q_3(A_1)}{Q_3(A_2)} \div \frac{Q_4(A_1)}{Q_4(A_2)} = \frac{\bar{x}_1\bar{x}_3}{\bar{x}_1\bar{x}_4} \div \frac{\bar{x}_2\bar{x}_3}{\bar{x}_2\bar{x}_4} = (\bar{x}_1\bar{x}_2\bar{x}_3\bar{x}_4) = c.$$

FUNCTIONS OF FINITE BAIRE TYPE

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The function space technique has been used, over the past 40 years, to obtain simple proofs of a great variety of facts regarding real and complex functions. In this note, the method is applied to the functions of finite Baire type to show that (a) there are sequences of functions of finite Baire type which converge uniformly to functions not of finite Baire type, but that (b) there are functions of Baire type ω_0 which are not the limits of any uniformly convergent sequences of finite Baire type.

The definition given in Hausdorff's book, [1], of functions of Baire type η, η a limiting ordinal, (different from the definition used here) indicates that he may have been aware of these facts. Nevertheless, the proof given here has a certain simplicity and appeal.

All functions considered are bounded real functions on the closed interval $[0, 1]$. Let B_0 be the set of continuous functions and, for every $\xi<\omega_1$, let B_ξ consist of all bounded functions that are limits of sequences of functions belonging to $\bigcup_{\nu<\xi} B_\nu$. The functions in B_ξ are the functions of Baire type ξ. Let C_ξ be the functions of Baire type ξ which are not of lower Baire type. The following are well known:

(a) For every $\xi<\omega_1$, C_ξ is nonempty.

(b) For every $\xi<\omega_1$, the limit of every uniformly convergent sequence of functions in B_ξ is itself in B_ξ.

(c) If $a_0=0<a_1< \cdots <a_n=1$, and the function f_i, defined on $[a_{i-1}, a_i)$, is of finite Baire type, $i=1, \cdots, n$, then the function, f, defined on $[0, 1]$ as $f=f_i$ on $[a_{i-1}, a_i)$, $f(1)=0$, is also of finite Baire type.

The proofs of (a), (b), (c) may be found in [1] and (b), (c) in [2]. The notations B_ξ, C_ξ will also denote the above classes of functions defined on subintervals of $[0, 1]$. The domain will always be clear from the context.

THEOREM 1. *There is a uniformly convergent sequence in $B=\bigcup_{n<\omega_0} B_n$ whose limit belongs to C_{ω_0}.*

Proof. B is a vector space. For every $f\in B$, let

$$\|f\| = \sup_{t\in[0,1]} |f(t)| .$$

Then B is a normed vector space. Now (b) says that, for every n, B_n is a Banach space. In particular, B_n is a closed proper subspace of B and so is nowhere dense in B. Hence $B=\bigcup_{n<\omega_0} B_n$ is of the first category and so is not complete. Thus, there are sequences in B which converge uniformly to functions of Baire type ω_0.

The following lemma is needed for the proof of Theorem 2.

LEMMA 1. *If E is a normed vector space and F is a closed vector subspace of E, there is an $x\in E$ such that $\|x\|=1$ and $\|x-y\|>\frac{1}{2}$ for every $y\in F$.*

For the proof, see [3].

THEOREM 2. *There is an $f \in C_{\omega_0}$ such that f is not the limit of any uniformly convergent sequence in B.*

Proof. Let $a_0 = 0 < a_1 < \cdots < a_n < \cdots$, $\lim_{n\to\infty} a_n = 1$. For every n, let f_n be a function defined on $[a_{n-1}, a_n]$, f_n of type C_n such that $\|f_n\| = 1$ and $\|f_n - f\| > \frac{1}{2}$ for every $f \in B_{n-1}$. For every n, define g_n such that $g_n = f_1$ on $[a_0, a_1)$, $g_n = f_2$ on $[a_1, a_2)$, $\cdots$, $g_n = f_n$ on $[a_{n-1}, a_n)$, and $g_n = 0$ on $[a_n, 1]$. By (c), $g_n \in B$. The sequence g_n converges to a function $g \in B_{\omega_0}$. But $\|g - f\| > \frac{1}{2}$ for every $f \in B$. Hence g is not the limit of any uniformly convergent sequence in B.

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ON THE EQUIVALENCE OF TWO SURFACES

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One of the fundamental classification theorems for surfaces states that any surface is topologically equivalent to a surface with at most two cross-caps.* This result rests on the fact that the sphere with three cross-caps is equivalent to the sphere with one cross-cap and a handle, or what is an equivalent statement, that *a Klein bottle with a boundary matched with the boundary of a cross-cap is topologically equivalent to a torus with a boundary matched with the boundary of a cross-cap.* The latter result is also rather interesting in view of the fact that the Klein bottle and torus themselves are *not* equivalent surfaces. The standard proofs of the italicized statement involve so-called "cutting and pasting" arguments, and the role of the cross-cap in the equivalence is obscured. Below is given a short informal proof which avoids cutting and pasting and which clarifies the "catalytic" role of the cross-cap.

We begin with the surface K, a Klein bottle with a boundary matched with the boundary of a cross-cap (Fig. 1). We shall show that K can be continuously

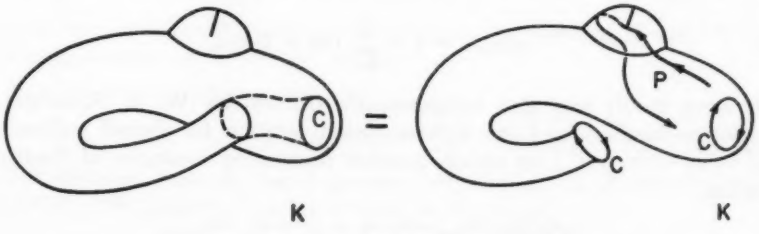

FIG. 1

* Due to A. W. Tucker; cf. S. Lefschetz, Introduction to Topology, Princeton, 1949, p. 85.

deformed into a torus with a boundary matched with the boundary of a cross-cap.

Let C be the simple closed curve on K at which the neck joins the mouth of the Klein bottle, and give C an orientation as shown. Consider a path P on K from the junction C up to the cross-cap, through the self-intersection, around the back of the cross-cap, and down again to the front of K. If C is moved continuously along this orientation-reversing path, then having traversed P, C will have the reverse of its initial orientation. Hence in the resulting surface T, (Fig. 2) the neck of the bottle will meet the surface at C without having to pass through T. Since topological equivalence is preserved in deforming C continu-

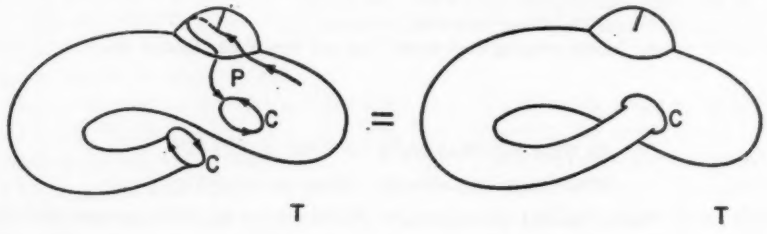

FIG. 2

ously along P, the surfaces K and T are equivalent. And as T is just a torus with a boundary matched with the boundary of a cross-cap, the proof is complete.

Notice that the important property of the cross-cap in the equivalence is its nonorientability. In particular, the italicized statement holds if the cross-cap is replaced by any nonorientable surface with a boundary.

ON SOME OF THE CLASSICAL ORTHOGONAL POLYNOMIALS

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C. K. Chatterjee [2] gave an interesting result for Bessel polynomials [4], namely,

$$y_n y_{n+1} = 1 + \sum_{k=0}^{n} (2k+1)x y_k^2. \tag{1}$$

This same result was also independently derived by W. A. Al-Salam [1]. Chatterjee further used the differentiation relation for Bessel polynomials, $x^2 y_n' = (nx-1)y_n + y_{n-1}$ to obtain another interesting analogue of Rodrigues' formula,

$$(x^2(d/dx))^k(y_n e^{-1/x}x^{-n}) = x^{-(n-k)}e^{-1/x}y_{n-k}. \tag{2}$$

He also derived a corresponding result with the help of the relation $x^2 y_{n-1}' = y_n - (nx+1)y_{n-1}$.

In the present note, we present similar sets of results for all the classical orthogonal polynomials (except those of Gegenbauer and Jacobi, where the results become cumbersome). Along with them we give a new set of differentiation formulas. The notations are as in [3].

We follow the following plan:

(a) will contain the analogue of Al-Salam's and Chatterjee's result (1) and a determinantal representation.*

(b) will contain the analogue of Chatterjee's result (2).

(c) will stand for the particular cases of the following lemma of the author:

LEMMA. *If U is a function of x of class C^n, then*

$$(d^n/dx^n)(\exp U) = \exp U((d/dx) + U')\cdot 1, \tag{3}$$

where $U' = \partial U/\partial x$ and $((d/dx)+U')\cdot 1$ is to be taken in the iterative sense.

1. Legendre polynomials.

(a) This has a recurrence relation

$$(n+1)P_{n+1} - (2n+1)xP_n + nP_{n-1} = 0. \tag{4}$$

Following Chatterjee or Al-Salam, we get

$$(n+1)P_{n+1}P_n = \sum_{k=0}^{n} (-1)^{k-1}(2k+1)xP_k^2.$$

Now (4) shows that $P_n(x)$ may be written as a determinant $P_n(x) = ((-1)^n/n!)$ det (a_{ij}), where

$$\begin{aligned} a_{ii} &= -(2n-2i+1)x, && 1 \leqq i \leqq n; \\ a_{i,i+1} &= a_{i+1,i} = n-i, && 1 \leqq i \leqq n-1; \\ a_{ij} &= 0, && \text{otherwise.} \end{aligned} \tag{5}$$

(b) $P_n(x)$ satisfies the recurrence relation $(1-x^2)P_n' = nP_{n-1} - nxP_n$. This may be written

$$[(1-x^2)^{1/2}(d/dx)][(1-x^2)^{-n/2}P_n] = n(1-x^2)^{-(n-1)/2}P_{n-1}(x).$$

Iterating this expression K times $(1 \leqq K \leqq n)$ with the operator $(1-x^2)^{1/2}(d/dx)$ and writing it as $(1-x^2)^{1/2}(d/dx)^K$, we get

$$\begin{aligned} &[(1-x^2)^{1/2}(d/dx)]^K[(1-x^2)^{-n/2}P_n] \\ &\qquad = n(n-1)\cdots(n-K+1)(1-x^2)^{-(n-k)/2}P_{n-k}. \end{aligned} \tag{6}$$

(c) Rodrigues' formula for Legendre polynomials is $P_n(x) = (1/(2^n n!))\cdot(d/dx)^n(x^2-1)^n$. Taking $U = n \log (x^2-1)$ in (3), we get

$$P_n(x) = (1/(2^n n!))(x^2-1)^n[(d/dx) + (2nx)/(x^2-1)]^n \cdot 1. \tag{7}$$

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* The latter was suggested to me by Professor P. L. Bhatnagar.

We merely quote the results for the rest of the polynomials

2. Tchebycheff polynomials.

(a) Since $T_n-2xT_{n-1}+T_{n-2}=0$, we get

$$T_nT_{n+1} = 2x\sum_{k=0}^{n}(-1)^{n-k}T_k^2. \tag{8}$$

Also, $T_n(x)=(-1)^n \det(b_{ij})$, where

$$\begin{aligned} b_{11} &= -x; \quad b_{ii} = -2x, && 2 \leqq i \leqq n; \\ b_{i,i+1} &= b_{i+1,i} = 1, && 1 \leqq i \leqq n-1; \\ b_{ij} &= 0, && \text{otherwise.} \end{aligned} \tag{9}$$

(b) Since $(1-x^2)T_n' = nT_{n-1}-nxT_n$, we get

$$\begin{aligned} &[(1-x^2)^{3/2}(d/dx)]^K[(1-x^2)^{-n/2}T_n] \\ &\qquad = n(n-1)\cdots(n-K+1)(1-x^2)^{-(n-K)/2}T_{n-k}, \end{aligned} \tag{10}$$

$$T_n(x) = (2^n n!)/(2n)!(x^2-1)^n[(d/dx)+(2n-1)x)/(x^2-1)]^n\cdot 1. \tag{11}$$

3. Hermite polynomials.

(a) Here we get

$$H_nH_{n+1} = 2x\sum_{k=0}^{n}(-1)^k 2^k[n!/(n-k)!]H_{n-k}^2. \tag{12}$$

4. Laguerre polynomials.

(a) We have

$$(n+1)L_nL_{n+1} = \sum_{k=0}^{n}(-1)^{n-k}(2k+1-x)L_k^2 \tag{13}$$

and $L_n(x)=((-1)^n/n!)\det(c_{ij})$, where

$$\begin{aligned} c_{ii} &= x-2n+2i-1, && 1 \leqq i \leqq n; \\ c_{i,i+1} &= c_{i+1,i} = n-i && 1 \leqq i \leqq n-1; \\ c_{ij} &= 0, && \text{otherwise.} \end{aligned} \tag{14}$$

5. Bessel polynomials.

(a) See Chatterjee or Al-Salam. We have $y_n(x)=\det(d_{ij})$, where

$$\begin{aligned} d_{11} &= 1+x, \quad d_{ii} = (2i-1)x, && 2 \leqq i \leqq n; \\ d_{i,i+1} &= -1, \quad d_{i+1,i} = 1, && 1 \leqq i \leqq n-1; \\ d_{ij} &= 0, && \text{otherwise.} \end{aligned} \tag{15}$$

(b) See Chatterjee [2].

(c) We get

$$y_n(x) = (x^{2n}/2^n)[(d/dx) + (2(1 + nx)/x^2)]^n \cdot 1. \tag{16}$$

It may be of interest to point out here that in the Sections 1, 2, 4, the formulas under (b) when $K = n$, give $n!$

I wish to express my thankfulness to Professors R. S. Krishnan and P. L. Bhatnagar for their encouragement and interest in this work. Thanks are also due to the referee for his valuable suggestions.

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ON THE NUMBER OF ZEROS OF A CUBIC RECURRENCE

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The cubic recurrence $(T) = (T_0, T_1, \cdots)$ is defined by the real numbers T_0, T_1, T_2 and the relation

$$T_{n+3} = PT_{n+2} - QT_{n+1} + RT_n \qquad (n = 0, 1, \cdots)$$

where P, Q, and R are real numbers. When $T_k = 0$, k is said to be a zero of (T). If any term of (T) is zero, it is no restriction to assume $T_0 = 0$.

M. F. Smiley [1] recently proved that if $(T) \neq (0, 0, 0, \cdots)$ and if the zeros of the companion of (T),

$$f(z) = z^3 - Pz^2 + Qz - R,$$

are nonvanishing real numbers with distinct absolute values, then (T) has no more than 3 zeros. This note extends these results as follows:

THEOREM. *A cubic recurrence* (T), *distinct from* (T'): $T_n' = Uu^n[1-(-1)^n]$, $n = 0, 1, \cdots$, *can have no more than three zeros when the zeros of its companion are nonvanishing real numbers.*

In view of Smiley's results it suffices to prove the theorem when the zeros of the companion of (T), u, v, and w, are such that

(1) $u = -v$; $|u| \neq |w|$,
(2) $u = v \neq w$, or
(3) $u = v = w$.

Case 1. $u = -v$; $|u| \neq |w|$. The term T_n may be expressed as $T_n = Uu^n + V(-u)^n + Ww^n$. Thus $T_0 = T_j = 0$ if and only if $U + V + W = 0$ and $Uu^j + V(-u)^j + Ww^j = 0$. Since $W = 0$ yields $U = -V$ and $T_n = Uu^n[1-(-1)^n]$, i.e. $(T) = (T')$, we may assume $W \neq 0$. If j is even we have $u^j = w^j$, $|u| = |w|$, contrary to $|u| \neq |w|$. Hence j is odd. If (T) has 3 zeros $0 < j < k$ then j and k are

odd and a suitable translate of (T) has zeros 0 and $k-j$ with $k-j$ even. We have just shown that this is impossible. Thus, in this case, (T) has at most two zeros.

Case 2. $u=v\neq w$. We have $T_n=(U+nV)u^n+Ww^n$, as well as $V\neq 0$, since $V=0$ and $T_0=T_j=0$ gives $(T)=(T')$. If $W=0$, the expression for T_n reduces to $T_n=nVn^n$ when $T_0=0$ and, in this case, (T) has exactly one zero. Now, $T_0=0$ if and only if $U+W=0$; hence, for $W\neq 0$, $T_n=0$ if and only if

$$1+nVU^{-1}=z^n, \tag{1}$$

where $z=w/u$. If z is positive, then (1) has at most one solution $n>0$. If z is negative, then (1) has at most two positive solutions. Consequently, (T) has at most three zeros when $u=v\neq w$.

Case 3. $u=v=w$. We have $T_n=(U+nV+n^2W)u^n$. If $T_0=0$ then $U=0$ and $T_n=0$, for $n>0$, if and only if $V+nW=0$. This is possible for at most one value of n. Consequently, (T) has at most two zeros in this final case.

Remark. Keeping the assumption that u, v, w are real, it is possible to determine those (T) with more than three zeros even if one or more of u, v, w is zero. Besides the (T') recurrences, one obtains those (T) such that $T_k=0$ for $k\geqq 3$.

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2. Morgan Ward, On the number of vanishing terms in an integral cubic recurrence, this MONTHLY, vol. 62, 1955, pp. 155–160.

CLASSROOM NOTES

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PLOTTING BOOLEAN FUNCTIONS

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By deforming the conventional truth table of logic so as to apply it to Boolean functions and displaying it graphically, one secures certain advantages, such as greater visual comprehension of the nature of the functions and a larger scope of applicability for the truth table.

Consider the following Boolean function of four variables and six products in disjunctive normal form, where $+$ denotes the Boolean sum, the immediate juxtaposition of elements designates the Boolean product, and $\bar{x}$ is the complement of x:

$$F=xyz+\bar{x}\bar{y}z+xy\bar{z}+\bar{x}y\bar{z}+\bar{x}yzw+\bar{x}\bar{y}\bar{z}\bar{w}.$$

Now construct a reference framework as shown in Figure 1. The first or top horizontal line is assigned to the variable x. This line is divided into two equal

segments, the heavy segment for the variable x itself, and the light segment for its complement $\bar{x}$. The second line, which has four equal segments, is assigned to the variable y; the heavy portions stand for y and the light portions stand for $\bar{y}$. The third line is assigned to the variable z and the fourth line is assigned to the variable w. The segments are to be distributed in the manner exhibited in Figure 1. With this distribution, the segments divide the abscissa into sixteen

FIG. 1

compartments, which correspond to the possible sets of values for the four variables (2^4).

Now take the first product xyz in F. Under the segment for x and, in turn, under the segment for y, place marks (X's) in the two comparments under z. Proceed in a like manner for the other products in F. The completed table is shown in Figure 2.

FIG. 2

The following procedure may be used for plotting functions in conjunctive normal form. Consider a function of four variables and five sums in conjunctive normal form. Let it be

$$G = (x + z + w)(\bar{x} + \bar{y} + z)(z + \bar{w})(x + \bar{y} + w)(x + y + z).$$

Using the laws of duality, invert G to get

$$\overline{G} = \bar{x}\bar{z}\bar{w} + xy\bar{z} + \bar{z}w + \bar{x}y\bar{w} + \bar{x}\bar{y}\bar{z}.$$

Now we may use the framework described above; but, in dealing with $\overline{G}$, we place marks in every compartment except those designated by the products in $\overline{G}$. The resulting plot is (by duality) the table for G. Figure 3 shows the table for the function G.

By combining the two techniques just described, one can form the table for a function such as

$$H = xyz + (x + y)(y + w) + x\bar{y}z + (\bar{x} + y + w)(x + \bar{w}).$$

FIG. 3

In order to unite the procedures, it is helpful to display the function as $H=H_1+H_2+H_3$, where $H_1=xyz+x\bar{y}z$, $H_2=(x+y)(y+w)$, and $H_3=(\bar{x}+y+w)(x+\bar{w})$. Now take separate, successive steps in accord with the techniques above to plot H_1, H_2, and H_3, always adding X's when authorized to do so but never removing X's which were entered by a previous, separate step. The table for this function appears in Figure 4.

FIG. 4

These simple tables for functions in either disjunctive normal form or conjunctive normal form or mixed form can be very quickly produced. The mixed forms amenable to the procedure are ones where, in a function $(A_1+\cdots+A_k)$, the A_i are products $(B_1\cdots B_m)$ and the B_j are sums of variables and their complements. The technique of analysis presented does not require the construction of a column of values for every subfunction in a given function and then, finally, a column of values for the function itself. The procedure can be used for functions of any number of variables—at least to that point where the framework becomes so large as to be cumbersome. The referee has suggested the name "Boole table" for the array described here; for this suggestion and others the writer is grateful.

The author has found the Boole table to be useful (1) as a graphical technique for simplifying Boolean functions, (2) as an alternative, under proper interpretation, to the Venn diagrams in the analysis of the categorical syllogism, and (3) as a procedure for determining whether Boolean functions are tautological, contingent, or contradictory. Under point (3) it may be observed that, due to the isomorphism of Boolean algebra and the propositional calculus, the Boole table provides a method for testing the validity of argument forms. The respective uses of the table for these cases should be clear.

The Boole table may be generalized. It is possible, as the referee has indicated, to isomorphically transform all the usual steps for truth tables into Boolean equivalents. We have, then, a perfectly general method for analyzing any Boolean function. The writer will present this generalization and some related ideas in a subsequent paper.

A REMARK ON LINEAR DIFFERENCE EQUATIONS

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It is well known that the order of the linear difference equation,

$$\sum_{j=0}^{n} p_j(k)\Delta^j y(k) = R(k), \qquad p_n(k) \equiv 1, \tag{1}$$

is not necessarily n; it may be less. If we express the differences in terms of the functional values, $y(k)$, $y(k+1)$, $\cdots$, $y(k+n)$, the order of (1) is then defined to be the difference between the largest and smallest arguments involved. We present now a simple test for the order of (1) which can be applied without resorting to formal decompositions of the differences to functional values. Since

$$\Delta^j y(k) = \sum_{i=0}^{j} (-1)^i \binom{j}{i} y(k+j-i),$$

we have

$$\sum_{j=0}^{n} p_j(k)\Delta^j y(k) = \sum_{j=0}^{n} p_j(k) \sum_{i=0}^{j} (-1)^i \binom{j}{i} y(k+j-i) = \sum_{r=0}^{n} B_r(k) y(k+r),$$

where $B_r(k) = \sum_{j=r}^{n} (-1)^{j-r} \binom{j}{r} p_j(k)$, $0 \leqq r \leqq n$. Thus, (1) is of order n if $B_0(k) \not\equiv 0$. If $B_0(k) \equiv 0$, then (1) is of order $n - r_0$ if $B_{r_0}(k) \not\equiv 0$ and $B_r(k) \equiv 0$, $0 \leqq r \leqq r_0 - 1$.

Example. Suppose $p_j(k) = \binom{n}{j}$, $j = 0, 1, \cdots, n$. Then

$$B_r(k) = \binom{n}{r} \sum_{j=r}^{n} (-1)^{j-r} \binom{n-r}{j-r} = \binom{n}{r} \sum_{s=0}^{n-r} (-1)^s \binom{n-r}{s}.$$

If $r = n$, $B_n(k) = 1$; if $0 \leqq r \leqq n-1$, $B_r(k) \equiv 0$, since $B_r(k)$ is then the binomial expansion of $(1-1)^{n-r}$. Thus (1) is of order zero and reduces to $y(k+n) = R(k)$. In terms of symbolic operators, we have the formula,

$$y(k+n) \equiv E^n y(k) \equiv (1+\Delta)^n y(k) = \sum_{j=0}^{n} \binom{n}{j} \Delta^j y(k),$$

which is readily established by mathematical induction.

A NOTE ON DIAGONABLE MATRIX POLYNOMIALS

D. W. ROBINSON, Brigham Young University

This note provides a necessary and sufficient condition for a matrix polynomial to be diagonable. Although the author is not aware of any mention of the result in the literature, it is doubtful that it has not previously been noticed. However, the result does provide an interesting classroom exercise in the use of principal idempotents of a matrix.

The author is indebted to the referee for several helpful suggestions.

THEOREM. *Let A be a square matrix over a field F in which the minimum polynomial $\prod_{i=1}^{r}(x-\alpha_i)^{s_i}$ of A is completely reducible. Let $\phi(x)$ be any polynomial in $F[x]$, and denote by $\sum_k \alpha_{ik}(x-\alpha_i)^k$ the expansion of $\phi(x)$ about α_i. Then $\phi(A)$ is diagonable if and only if $\alpha_{ik}=0$ whenever $0<k<s_i$.*

Proof. Let $E_1, \cdots, E_r$ be the principal idempotent elements of A corresponding, respectively, to the distinct eigenvalues $\alpha_1, \cdots, \alpha_r$. Since $\alpha_{i0}=\phi(\alpha_i)$ and $E_i(A-\alpha_i I)$ is nilpotent of index s_i, then

$$E_i(\phi(A)-\phi(\alpha_i)I)=E_i\sum_{k=1}^{s_i-1}\alpha_{ik}(A-\alpha_i I)^k. \tag{1}$$

Also, let $F_i=\sum E_j$, where the sum is taken over all j such that $\phi(\alpha_j)=\phi(\alpha_i)$. Since the principal idempotents of A are pairwise orthogonal, and each commutes with A, then, by (1), $F_i(\phi(A)-\phi(\alpha_i)I)$ is nilpotent of index, say, t_i. Therefore, since $\sum F_i=I$, where the sum is over all distinct F_i, and $F_i\neq 0$, it is evident that $\prod(x-\phi(\alpha_i))^{t_i}$ is the minimum polynomial of $\phi(A)$, where the product extends over all distinct factors $x-\phi(\alpha_i)$. Now, a matrix is diagonable if and only if its minimum polynomial factors into a product of distinct linear factors. Hence, if $\alpha_{ik}=0$ whenever $0<k<s_i$, then each $t_i=1$ and $\phi(A)$ is diagonable. Conversely, since (1) is nilpotent of index at most t_i, if $\phi(A)$ is diagonable, then $t_i=1$ and (1) is the zero matrix. Thus, since the system $E_i(A-\alpha_i I), \cdots, E_i(A-\alpha_i I)^{s_i-1}$ is linearly independent, $\alpha_{ik}=0$ whenever $0<k<s_i$.

COROLLARY. *Let the notation and conditions be as in the theorem and suppose, furthermore, that F is of characteristic zero. Then $\phi(A)$ is diagonable if and only if $\phi^{(k)}(\alpha_i)=0$ whenever $0<k<s_i$.*

Proof. The conclusion is an immediate consequence of the fact that $\phi^{(k)}(\alpha_i)=k!\alpha_{ik}$.

This note is prompted by a theorem in a recent paper [1] of L. Mirsky and R. Rado. Since their result is an immediate corollary of the above theorem, it is also included here. It is first observed, however, that Mirsky and Rado stated their result only for the field of complex numbers.

COROLLARY. *Let the notation and conditions be as in the theorem. Furthermore, let $f(x)$ be a polynomial in $F[x]$ such that $f(A)=0$, and suppose that $\phi'(\alpha_i)\neq 0$ whenever $f'(\alpha_i)=0$. Then, if $\phi(A)$ is diagonable, then A is diagonable.*

Proof. Recalling that the minimum polynomial of A must divide any such polynomial $f(x)$, it follows that $\phi'(\alpha_i)=\alpha_{i1}\neq 0$ whenever $s_i>1$. Hence, if $\phi(A)$ is diagonable, then each $s_i=1$ and A is also diagonable.

Reference

1. L. Mirsky and R. Rado, A note on matrix polynomials, Quart. J. Math. Oxford Ser. (2), vol. 8, 1957, pp. 128–132, Theorem 1(i).

A PROBABILITY RELATED TO THE EUCLIDEAN ALGORITHM

H. S. THURSTON, University of Alabama

If α and $\beta \neq 0$ are integers of the quadratic field $\mathrm{Ra}(\sqrt{m})$ where m is a square-free, negative, rational integer, what is the probability that integers μ and ν of the field can be found such that $\alpha = \beta\mu + \nu$ where $N(\nu) < N(\beta)$? This question arose in a course in the theory of algebraic numbers and its solution was found to involve an interesting bit of integral calculus.

In the complex plane, the points representing the integers of $\mathrm{Ra}(\sqrt{m})$ are located at the vertices of *cells*, parallelograms whose bases are of length one unit and whose sides are inclined at an angle ϕ. If $m \equiv 2$ or $3 \pmod 4$, the height of the cell is $\sqrt{(-m)}$ and $\phi = \pi/2$. If $m \equiv 1 \pmod 4$ the height is $\frac{1}{2}\sqrt{(-m)}$ and $\phi = \mathrm{Arctan}\ \sqrt{(-m)}$. The existence of μ and ν requires that the point $z = \alpha/\beta$ shall be within at least one of the unit circles with center at a vertex of a cell. Thus the required probability is that percentage of the total area of a cell which is covered by such circles.

As is well known, if m is -1, -2, -3, -7 or -11, the probability is unity. For other negative values of m, let us consider the cell $ABCD$ with vertices $A(0, 0)$ and $B(1, 0)$, and lying in the first quadrant (Figs. 1, 2). The circles $r = 1$ and $r = 2\cos\theta$ cut AD and BC, respectively, at E and F, and intersect at G on

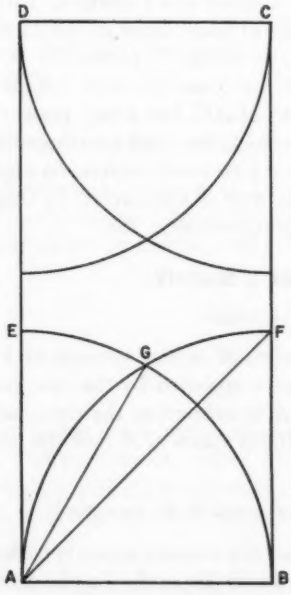

FIG. 1: $m = -5$.

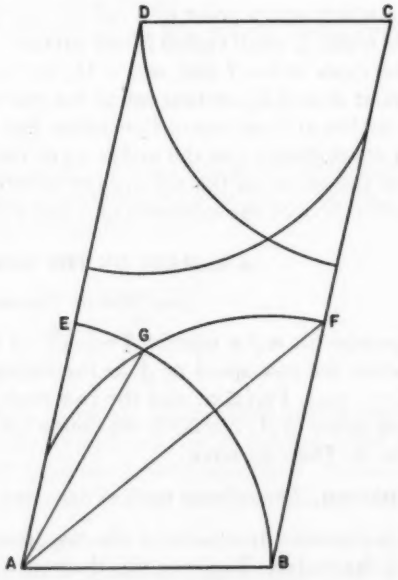

FIG. 2: $m = -19$.

the line $\theta=\pi/3$. The area inside the cell and covered by these circles is made up of the area A_1 of the triangle ABF, the area A_2 of the region FAG bounded by the lines $\theta=\phi/2$, $\theta=\pi/3$, and the circle $r=2\cos\theta$, and the area A_3 of the sector GAE. These areas are given as follows:

$$A_1=\tfrac{1}{2}\sin\phi,\quad A_2=2\int_{\phi/2}^{\pi/3}\cos^2\theta d\theta,\quad\text{and}\quad A_3=\frac{1}{2}\int_{\pi/3}^{\phi}d\theta.$$

If $m\equiv 2$ or 3 (mod 4), $\phi=\pi/2$, the area is obviously independent of m, and is readily shown to be $(2\pi+3\sqrt{3})/12$. If $m\equiv 1$ (mod 4) the areas are

$$A_1=\tfrac{1}{2}\sin\phi,\quad A_2=\pi/3-\phi/2+\sqrt{3}/4-\tfrac{1}{2}\sin\phi,\quad A_3=\phi/2-\pi/6,$$

where $\phi=\text{Arctan}\,\sqrt{(-m)}$, and the total area is again $(2\pi+3\sqrt{3})/12$. Since the circles with centers at C and D cover an area equal to this, and since the areas of the cells in the two cases are, respectively, $\sqrt{(-m)}$ and $\frac{1}{2}\sqrt{(-m)}$, the probability of the existence of μ and ν is

$$p=(2\pi+3\sqrt{3})/6\sqrt{(-m)}\ \text{ if }\ m\equiv 2\text{ or }3\ (\text{mod }4)\quad\text{and}\quad m\leqq -5$$

and

$$p=(2\pi+3\sqrt{3})/3\sqrt{(-m)}\ \text{ if }\ m\equiv 1\ (\text{mod }4)\quad\text{and}\quad m\leqq -15.$$

It will be instructive for the students to draw figures for $m=-3$, -7, -11, and -15. These will show the gradual evolution, when $m\equiv 1$ (mod 4), from the case in which every point of a cell is covered by at least three of the circles to that in which a small region is not covered by any circle. Of particular interest are the cases $m=-7$ and $m=-15$. In the former case the unit circles with centers at A and C are tangent at the mid-point of AC, but every point of the cell is within at least one of the circles. For $m=-15$, the circles with centers at B and D are tangent at the midpoint of BD, and two small regions on opposite sides of the center of the cell are not covered by any of the circles. In this case the probability of the existence of μ and ν is approximately .988.

A REMARK ON THE RANK OF A MATRIX

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Consider an $m\times n$ matrix $A=\|a_{ij}\|$. It consists of m row vectors of $V_n(F)$. We define the row space of A as the vector space spanned by the row vectors $(a_{i1},\cdots,a_{in})$, $1\leqq i\leqq m$, and the row rank of A is defined as the dimension of the row space of A. Similarly we define the column space of A and the column rank of A. Then we have

THEOREM. *The column rank of any matrix is equal to its row rank.*

This common dimension of the row space and the column space is called the rank of the matrix. To prove this theorem and to find the rank of a given matrix A is a starting point of the theory of matrices. I think that the most appropriate

treatment of this subject in our classroom may be probably as follows.

Apply the elementary row operations and bring A to the row-reduced echelon form, such as the form of the following matrix:

$$\begin{vmatrix} 1 & d_{12} & 0 & d_{14} & 0 & d_{16} \\ 0 & 0 & 1 & d_{24} & 0 & d_{26} \\ 0 & 0 & 0 & 0 & 1 & d_{36} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{vmatrix}.$$

As readily proved, we have

LEMMA 1. *The elementary row operations do not alter the row space of the matrix.*

Therefore, the row rank of any matrix is equal to the row rank of its row-reduced echelon form. But obviously the row rank of the row-reduced echelon matrix is equal to the number of its nonzero rows. Hence the row rank of A is equal to the number of the nonzero rows of its row-reduced form.

So far the analysis is essentially that in [1], but we now extend the ideas. It needs only a few steps.

First, we see at a glance that the dimension of the column space of the row-reduced echelon matrix is equal to the number of columns containing the entry 1 somewhere and only zeros elsewhere. Therefore, about the row-reduced echelon matrix, the column rank and the row rank are equal.

Next, consider the change of the column space due to the elementary row operations.

LEMMA 2. *By any elementary row operation the column space of a matrix is mapped isomorphically onto the new column space.*

For example, consider an interchange of two rows, the first and second row, and an addition of c times the first row to the second row. Then all vectors of the column space are mapped respectively as follows:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \cdot \\ \cdot \\ x_m \end{pmatrix} \rightarrow \begin{pmatrix} x_2 \\ x_1 \\ x_3 \\ \cdot \\ \cdot \\ x_m \end{pmatrix}, \qquad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \cdot \\ \cdot \\ x_m \end{pmatrix} \rightarrow \begin{pmatrix} x_1 \\ x_2+cx_1 \\ x_3 \\ \cdot \\ \cdot \\ x_m \end{pmatrix}$$

Each of these mappings is readily seen to be an isomorphism respectively between the two vector spaces. Hence by elementary row operations, even if the column space might be changed, its dimension remains unaltered. It is equal to the rank of the row-reduced echelon form. Thus we have proved the theorem.

Reference

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MATHEMATICAL EDUCATION NOTES

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THE SYRACUSE UNIVERSITY "MADISON PROJECT"

ROBERT B. DAVIS, Syracuse University

This is an extremely brief report on the Syracuse University "Madison Project." Longer accounts appear in the M.I.T. *Technology Review* and elsewhere (see references).

The project is concerned with mathematics teaching, particularly in grades 3–9 (but also, to a lesser extent, with the process of learning mathematics on the college and graduate levels). It operates mainly in New York State and in Connecticut. (The name, incidentally, derives from the name of the first school in which the project operated, Madison Junior High School in Syracuse, New York.)

The main objective of the project is a behavioral study of how people learn mathematics. A two-year program in algebra, suitable for presentation to children beginning as early as grade 3, is one concrete result of the project's studies. A workbook to accompany this algebra course is another concrete by-product of project research.

The workbook. The workbook provides for a two-year sequence in algebra, with the following features:

(i) An unusual effort is made to present basic mathematical concepts clearly, and to keep the logic as nearly 100% honest as possible. This latter requirement implies that most of the logic is inductive, by generalization from experience, but a small amount of very simple deductive logic is included (*e.g.*, one of the rare pieces of deductive logic—if a single substitution into an open sentence results in a false statement, the open sentence was not an *identity*, because for identities *every* substitution produces a true statement).

(ii) The language itself is kept casual and unobtrusive. Mathematical concepts are developed from experience with mathematical situations, not by exposition.

(iii) The student learns mainly from his experiences in analyzing a sequence of problems. For example, by plotting linear functions by trial and error, he discovers how to identify and make use of the slope coefficient. We know from experience that even low-I.Q. children can readily discover these things for themselves, if we merely give them an appropriate sequence of problems. *The course includes extremely little exposition*—the workbook itself is mainly a sequence of questions, such as: (1) Graph $y=2x+1$; (2) Graph $y=3x+1$; (3) Graph $y=5x+4$; etc.

(iv) This course has been taught with notable success to gifted fourth and

fifth graders, to low-I.Q. 7th graders, and to many other groups, beginning as early as grade 3, and including a full range of I.Q.'s from low-I.Q. classes to gifted classes.

(v) The workbook deals only with algebra. It can be used as occasional enrichment material: a fourth-grade teacher, for example, might introduce some one or two topics from the workbook. In this way, both teachers and school systems can introduce more algebra into the grades by a gradual process. This flexibility is a big advantage over a closely integrated program which might have to be adopted as a whole or else not at all.

(vi) The program lays great stress on identifying the basic ideas (and especially the *hard* ideas) of mathematics and introducing them *early*, so that they may be progressively refined in many subsequent years of study. The project advocates a first appearance of the concept of limit, for example, as early as grade 4, in simple form. Specifically, we can seek numbers which, when multiplied by themselves, result in a product of 2. The germ of the entire concept of limit is contained in this problem, and can be brought out in subsequent study during the next six or eight years.

(vii) Since mathematics, we believe, consists in large measure of mastery of the art of creative problem solving, students practice this at every step of their study. This is a corollary of our nonexpositional approach of teaching by asking a sequence of questions.

Why algebra in the grades? We have many reasons for wishing to begin the study of algebra as early as grade 3 or 4. To list just a few:

(i) We believe a great deal of algebra is *easier* than a great deal of arithmetic.

(ii) We believe that many algebraic concepts and tools lie at the heart of arithmetic, and can be used both to give deeper insight into arithmetic and to simplify the study of arithmetic.

(iii) We believe that continual practice in creative problem solving is essential to the true nature of mathematics, and that algebra provides more (and easier) material for the practice of this art than arithmetic does.

Other aspects of the project. As mentioned above, the workbook is merely a concrete by-product of project research. We believe that problems in the teaching of mathematics should not be regarded as national crises to be overcome by gigantic crash programs. Rather, we insist that a wiser approach is to regard mathematics education as a uniquely fruitful field for studying the behavior of the human mind; curricula, courses, tests, and so on should appear as "applied" bonuses that stem from a program of basic research.

The experience of four years has convinced us that in working with the schools it is necessary to operate on a regional basis, with collaboration between teachers and university mathematicians. A program brought in from outside cannot and will not be installed successfully within a local school. If you want a better program in a school, you have to cultivate it and grow it right there on the spot.

The use of films. Many observers have been impressed by the project's success in getting fourth graders *to discover for themselves* the coefficient rules for quadratic equations, and so on. We feel this success is closely related to the conduct of our classes. This creates two problems: First, how can we show teachers just what we mean when we speak of "discovery method," "making mathematics an intellectual challenge rather than dreary routine," and so on? Second, how can we study project classes in order to determine why they *do* work so well? We hope that films (with sound) may provide an answer to both questions.

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United States Registry of Junior and Senior High School Science and Mathematics Teaching Personnel

The United States Registry of Junior and Senior High School Science and Mathematics Teaching Personnel has been made possible through grants by the National Science Foundation and the Future Scientists of America Foundation of NSTA to the National Council of Teachers of Mathematics and the National Science Teachers Association (NSTA). At present, the registry consists of more than 120,000 individuals and their school addresses on coded cards punched to indicate fields of teaching responsibility.

Information regarding the conditions under which this name list can be obtained for the distribution of professional type materials can be obtained for mathematics by writing to Myrl H. Ahrendt, Executive Secretary, National Council of Teachers of Mathematics, 1201 Sixteenth Street N.W., Washington 6, D. C.

The numbers of teachers in the registry are as follows: (Note—since many teachers have more than one subject responsibility, totals do not agree with the sum of the group items.)

Mathematics		76,600	Science		68,400
Grades 7–8	32,000		General Science	33,600	
Grades 9–12	48,600		Biology	21,800	
Dept. Heads	14,500		Chemistry	14,700	
Supervisors (State, County & Local)	308		Physics	12,500	
			Dept. Heads	12,700	
			Supervisors (State, County & Local)	339	

Advanced Mathematics by TV

The University of Notre Dame is expanding its experiment with closed circuit television as a medium for instruction in advanced mathematics.

Approximately 180 high school and college teachers, the majority of them nuns, enrolled for a 1959 summer course in "Higher Algebra and Analytic Geometry" via TV. Arnold Ross, head of Notre Dame's mathematics department conducted the course from the studios of WNDU-TV, the university-operated station. Except for a rotating studio audience of about 20, the students viewed his lectures on monitors set up in six classrooms of the Engineering Building two blocks away.

This experiment grew out of an experimental two-week series of lectures on "Boolean Algebra" offered by Ross via closed circuit TV in the summer of 1958. Although Ross was the only person appearing on the TV screen, he stressed that the course was not a one-man project. Supplementing his daily one-hour lectures were seminars conducted three times a week by other faculty members of the mathematics department. The seminar leaders worked closely with Dr. Ross in evaluating the students' progress.

—From a report which appeared in several Indiana newspapers and the Providence (R. I.) Journal.

NASDTEC Study of Certification Requirements for Teachers of Secondary-School Mathematics and Science

The American Association for the Advancement of Science has received a grant from the Carnegie Corporation of New York for a study of certification requirements for teachers of secondary-school science and mathematics. The grant is based on a proposal addressed to the Carnegie Corporation by the National Association of State Directors of Teacher Education and Certification (NASDTEC). Since this organization is unincorporated, the grant has been made to AAAS which will hold the funds and administer the grant in cooperation with NASDTEC. J. R. Mayor, Director of Education, AAAS, will serve as director of the certification study, and William Viall, formerly chief certification officer in New York state and immediate past president of NASDTEC, has been appointed associate director. E. G. Begle is a member of the advisory board for the study. The offices will be in the AAAS building, 1515 Massachusetts Avenue, N.W., Washington.

The study, planned as an eighteen months' enterprise, has the following goals:

1. To develop procedures whereby representatives of logically interested organizations and state certification and accreditation officials may work together effectively in the development of teacher preparation programs, and to prepare suggested guides for program approval by state certification officers.
2. To reach common agreements in regard to qualitative factors in the preparation and certification of teachers of science and mathematics.
3. To identify areas of instruction in science and mathematics for which preparation programs should be developed and requirements established.
4. To provide continuity in teacher-education programs in science and mathematics beyond the baccalaureate degree in order to facilitate the fulfillment of requests to teach these subjects in any state.

The study is planned to involve state action under which in each state there would be appointed a state committee under the Director of Teacher Education and Certification. The committee would include scientists as well as teachers and professional educators. Four regional groups will be designated to study materials available on teacher certification in science and mathematics, including those prepared by scientific groups and to draft what they conceive to be an ideal program. The result of the regional studies

will be referred to the states and eventually a national conference of certification officers and other interested persons will be held.

Among the purposes of the national conference would be the making of plans for the development of reciprocity agreements among the states. In the proposal it is pointed out that in quite a number of instances several states in a region have reciprocity agreements which affect candidates having certificates in the elementary field because agreements have been reached in general as to what constitutes a sound program for the preparation of elementary teachers. No such agreement exists for secondary level teachers nor seems possible without action of the type proposed in this document.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, University of Maine

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, University of Maine, Orono, Maine. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1401. *Proposed by Emil Grosswald, Institute for Advanced Study*

Prove that for every integer $k \geqq 1$,

$$\frac{d^{k-1}}{du^{k-1}}\left(\frac{1}{e^u-1}\right) = (-1)^{k-1}\sum_{r=1}^{k}\frac{\gamma_k(r)}{(e^u-1)^r},$$

where the constants $\gamma_k(r)$ are all positive and satisfy

$$\gamma_k(r) = \sum_{\nu=0}^{r-1}(-1)^\nu\binom{r-1}{\nu}(r-\nu)^{k-1}.$$

In particular, show that $\gamma_k(k) = (k-1)!$.

E 1402. *Proposed by F. Leuenberger, Zuoz, Switzerland*

Let r denote the radius of the inscribed sphere of a tetrahedron T and let r_i $(i=1, 2, 3, 4)$ denote the radii of the exspheres of T which touch one face of T and the other three faces of T produced. Show that $\sum_{i=1}^{4} r_i \geqq 8r$, with equality if and only if T is isosceles.

E 1403. *Proposed by T. J. Rivlin, IBM Corp., Yorktown Heights, N. Y.*

Let $p(x) = a_0 + a_1x + \cdots + a_nx^n$, with the sequence of coefficients a_3, a_5, a_7, $\cdots$, a_m $(m=n,\ n$ odd; $m=n-1,\ n$ even) either all nonnegative or all nonpositive. Show that:

(a) there exists a number y_n such that $[p(1)-p(-1)]/2=p'(y_n)$,
(b) if n is odd and >1, $|y_n| \leqq (1/n)^{1/(n-1)}$,
(c) if n is even and >2, $|y_n| \leqq [1/(n-1)]^{1/(n-2)}$,
(d) the bounds on $|y_n|$ are the best possible,
(e) when $n=1$, 2 we may take $y_n=0$.

E 1404. *Proposed by H. N. Ward, Harvard University*

(a) In Euclidean 2-dimensional space R^2, let L be the lattice of points with integral coordinates, and T any nondegenerate triangle (including its boundary) whose vertices belong to L. Let T' be any translation of $-T(=\{-x|x\in T\})$ which has members of L as vertices; that is, $T'=p-T$, where $p\in L$. Show that if $T\cap T'\neq\phi$, then $T\cap T'$ contains a member of L.

(b) Show that an analogous statement in Euclidean n-dimensional space R^n, $n>2$, where simplexes are used for triangles, fails.

E 1405. *Proposed by L. L. Garner, University of North Carolina*

Find (a) an expression for the general term, (b) the sum of the first n terms of the series $1+22+333+\cdots$.

SOLUTIONS

The 10^{1000} Digit in the Sequence of Natural Numbers

E 1371 [1959, 512]. *Proposed by G. C. Thompson, Security Mutual Life Ins. Co., Binghamton, N. Y.*

On p. 457 of Vol. I of Dickson's *History of the Theory of Numbers* appears the statement: "E. Barbier asked what is the 10^{1000}th digit written if the series of natural numbers be written down." What is this digit?

Solution by E. S. Eby, U. S. Navy Underwater Sound Laboratory, Fort Trumbull, New London, Conn. The total number of digits used to write all integers of less than $n+1$ digits is $S_n=[(9n-1)10^n+1]/9$; this can be verified by mathematical induction. Now $S_{997}<10^{1000}<S_{998}$. Hence the 10^{1000}th digit occurs in an integer of 998 digits, and, in fact, is the rth digit of $10^{997}+q$, where $10^{1000}-S_{997}=998q+r$, $0\leqq r<998$. Evaluation of q and r show the 10^{1000}th digit to be the 31st digit of the integer

$$100\ 31173\ 45802\ 71654\ 41995\ 10131\ 37385\cdots$$

so that the answer to E. Barbier's question is 3.

Also solved by Louis and Frances Bauer, John Brown, N. J. Fine, A. R. Hyde, John Jordan, Sidney Kravitz, O. W. Laing, W. L. McDaniel, D. C. B. Marsh, G. M. Roe, C. M. Sandwick, Sr., D. R. Sondergeld, and the proposer.

Not all the solvers found the answer 3.

Existence of a Derivative

E 1372 [1959, 512]. *Proposed by J. R. Munkres, Princeton University*

Let $f(x)$ be continuous on the closed interval $[0, 1]$ and have a derivative on the open interval $(0, 1)$. Suppose $|xf'(x)-f(x)+f(0)| < x^2M$ for all x on $(0, 1)$, where M is fixed. Does $f'(0)$ exist?

Solution by N. J. Fine, Institute for Advanced Study. The right-hand derivative exists. We may suppose that $f(0)=0$. Let $h(x)=f(x)/x$ for $0<x<1$. It is sufficient to show that $\lim_{x\to 0} h(x)$ exists. Now h is differentiable in $0<x<1$, and

$$f'(x) = [xh(x)]' = xh'(x) + h(x),$$

whence $xf'(x)-f(x)=x^2h'(x)$. By hypothesis, therefore,

$$|x^2h'(x)| < Mx^2, \text{ or } |h'(x)| < M \qquad (0 < x < 1).$$

If $\{x_n\}$ is any sequence in $(0, 1)$ tending to 0, then

$$h(x_m) - h(x_n) = h'(\xi)(x_m - x_n),$$

where ξ is between x_m and x_n, so $|h(x_m)-h(x_n)| < M|x_m-x_n| < \epsilon$, provided that $m, n > N(\epsilon)$. Thus $\{h(x_n)\}$ is a Cauchy sequence, and hence convergent. Since $\{x_n\}$ was arbitrary, $\lim_{x\to 0} h(x)$ exists.

Also solved by J. L. Brown, Jr., R. F. Brown and Joel Levy (jointly), E. S. Eby, H. B. Emerson, George Glauberman, L. C. Graue, Alfred Gray, J. Hooley, R. H. Hou, A. R. Hyde, D. C. B. Marsh, Q. G. Mohammad, D. L. Muench, D. S. Passman, W. A. Peterson, John Rainwater, W. A. Riley, J. T. Rosenbaum, E. C. Schlesinger, Jack Silver, J. V. Whittaker, and the proposer. Late solution by Sidney Glusman.

Editorial Note. The hypothesized continuity of $f(x)$ at $x=0$ is unnecessary.

A Power Series

E 1373 [1959, 512]. *Proposed by W. A. Veech, Dartmouth College*

Find the radius of convergence and the function $f(z)$ represented by

$$f(z) = \sum_{n=0}^{\infty} s_n z^n \quad \text{if} \quad s_n = \sum_{k=0}^{n} k\binom{n}{k}.$$

Solution by E. S. Eby, U. S. Navy Underwater Sound Laboratory, Fort Trumbull, New London, Conn. Since

$$s_n = \sum_{k=0}^{n} k\binom{n}{k} = \sum_{k=0}^{n-1} n\binom{n-1}{k} = n2^{n-1},$$

we have

$$f(z) = \sum_{n=0}^{\infty} s_n z^n = \sum_{n=1}^{\infty} n2^{n-1}z^n = z\sum_{n=0}^{\infty} (n+1)(2z)^n = z(1+2z)^{-2},$$

with radius of convergence $1/2$.

Also solved by Norman Bauman, A. P. Boblétt, Julian Braun, D. A. Breault, J. L. Brown, Jr., R. F. Brown and Joel Levy (jointly), A. E. Danese, H. B. Emerson, N. J. Fine, Michael Goldberg, L. D. Goldstone, George Glauberman, Cornelius Groenewoud, J. R. Hanne, J. C. Hickman, J. Hooley, R. H. Hou, A. R. Hyde, John Jordan, Morton Kupperman, T. V. Laxminavasimhan and K. R. Rajagopalan (jointly), W. L. McDonald, D. C. B. Marsh, Q. G. Mohammad, C. S. Ogilvy, F. R. Olson, D. S. Passman, John Rainwater, E. M. Scheuer, Jack Silver, Benjamin Sims, Robert Spira, Eric Sturley, J. R. Swenson, W. F. Trench, Chih-yi Wang, David Zeitlin, and the proposer.

Another Square-Covering Problem

E 1374 [1959, 513]. *Proposed by Yonder Page, Kitchawan, N. Y.*

What is the minimum d such that the unit square may be covered with five sets each having diameter d?

Solution by J. L. Selfridge, University of California at Los Angeles, and R. A. Willoughby, IBM Corp., Yorktown Heights, N. Y. Use a square of side 32 with corners at $(\pm 16, \pm 16)$. Draw appropriate polygons with corners at $(\pm 8, -16)$, $(\pm 8, -3)$, $(\pm 16, 3)$, $(0, 3)$, $(0, 16)$ to show that $d^2 \leqq 425$. Suppose d smaller. There are four corner sets. None covers 1/4 the perimeter. The fifth set touches some side, say the bottom. The fifth set covers $(0, 3)$. The lower corner sets cover $(\pm 8, -16)$ respectively. The upper corner sets cover $(\pm 16, 3)$. No set covers $(0, 16)$. Thus for the unit square $d = 5\sqrt{17}/32$. Note that the fifth set will not fit into a circle of diameter d.

Also solved by Norman Bauman, Julian Braun, J. W. Ellis, Michael Goldberg, P. C. Hammer, and O. Tannenbaum.

Editorial Note. See E 1311 [1958, 775].

Construction of a Triangle Given A, m_a, t_a

E 1375 [1959, 513]. *Proposed by L. D. Goldstone, N. Y. State Public Works Lab., Albany, N. Y.*

Construct a triangle given A, m_a, t_a, that is, given a vertex angle and the median and angle bisector issued from this vertex.

I. *Solution by Victor Thébault, Tennie, Sarthe, France.* Let BAC be the sought triangle whose angle A, angle bisector $AL = t_a$, and median $AM = m_a$ are given. Let B' be the reflection of B in A. Then CB' is parallel to AM and equal to $2m_a$, since M bisects BC. Through L draw the parallel to BA to cut CB' in L'. Since, on the one hand, AL is the bisector of angle A, and on the other hand, BB' and LL' are parallel, we have

$$AB'/AC = AB/AC = LB/LC = L'B'/L'C.$$

Comparison of the first and last of these ratios shows that AL' bisects angle CAB'. Triangle LAL' is then right-angled at A, and since we know a side $(AL = t_a)$ and an angle (angle $ALL' = A/2$) of the triangle, we can construct the triangle and obtain $AL' = t_a' = t_a \tan A/2$. In triangle $B'AC$ we then know a side $(CB' = 2m_a)$, the opposite angle $(= \pi - A)$, and the corresponding angle

bisector ($t_a' = AL'$). The construction of this triangle is classic: on $B'C = 2m_a$, describe the arc containing angle $\pi - A$; then angle bisector AL' passes through the midpoint I of the supplementary arc and

$$IA - IL' = t_a', \qquad (IA)(IL') = (IB')^2 = m_a^2/\sin^2 A/2,$$

whence (knowing their difference and their product) IA and IL' can be constructed. Triangle $B'AC$ being constructed, it suffices to find the symmetric of B in A in order to construct triangle BAC.

Note. This problem was proposed by Desmons (*Journal de Mathématiques Elémentaires et Spéciale de Longchamps*, 1881, p. 92, question 301) and solved with the use of calculus by Debray (*ibid*, p. 398). The preceding geometric solution, inspired by that of L. Bickart (*Journal de Vuibert*, 31e année, p. 61, question 6345), constitutes a geometric solution of the problem of finding the points of intersection of a conic and a circle centered at one of the vertices of the conic. An angle $xAy = A$ being given, as well as a point L on its bisector, it suffices to observe that Ax, Ay intercept on an arbitrary line through L a segment whose midpoint M describes a hyperbola h having vertices A and L and whose asymptotes are parallel to Ax, Ay. Problem E 1375 is that of finding the intersections of h and the circle of center A and radius m_a.

II. *Solution by Roscoe Woods, State University of Iowa.* Let ABC be the required triangle. Let the bisector (internal) of angle A intersect BC in U and let A' be the midpoint of BC. Let M denote the foot of the perpendicular from A' to AU. If $2\theta = A$, it is readily seen that $AM = [(b+c)\cos\theta]/2$. If segment AM can be constructed then point A' can be found, since $AA' = m_a$ is given. Consequently the line through U (known since $AU = t_a$) and A' determine the vertices B and C, since the angle A is given. From the figure it is obvious that in order to have a solution, m_a must be greater than or equal to t_a.

Since $4m_a^2 = 2b^2 + 2c^2 - a^2$ and $a^2 = b^2 + c^2 - 2bc\cos A$ we find, by use of the relation $2\sin^2\theta = 1 - \cos A$, that $4m_a^2 = (b+c)^2 - 4bc\sin^2\theta$. A short calculation shows that $t_a(b+c) = 2bc\cos\theta$. If bc is eliminated from these two expressions, a quadratic equation in $b+c$ is obtained. From the positive root of this equation an expression for AM is readily found, namely

$$AM = \tfrac{1}{2}(b+c)\cos\theta = \tfrac{1}{2}t_a\sin^2\theta + \sqrt{\left\{\left[\tfrac{1}{2}t_a\sin^2\theta\right]^2 + m_a^2\cos^2\theta\right\}}.$$

Segment AM may now be constructed. From the right angle RST, with angle $R = \theta$ and hypotenuse $RT = t_a$, the projection of ST upon RT has length $t_a\sin^2\theta$. Also, from a right triangle with hypotenuse m_a and one acute angle θ, $m_a\cos\theta$ may be found. The details of the construction of AM may be omitted since they are obvious. There are two solutions if m_a is greater than AM, one solution when $AM = m_a (= t_a)$, and no solution when m_a is less than AM.

Also solved by Josef Langr, D. C. B. Marsh, Beckham Martin, and the proposer. N. A. Court furnished some remarks.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers, The State University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers, The State University, New Brunswick, New Jersey. All manuscripts should be typewritten with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should enclose any solutions or information that will assist the editor. In general, problems in well-known textbooks or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4877 [1959, 921] Corrected. *Proposed by O. P. Aggarwal and Irwin Guttman, University of Alberta.*

Show that $\int_0^a \exp(-\frac{1}{2}t^2)\,dt = (\frac{1}{2}\pi - aI)^{1/2}$, where

$$I = \int_{a^2}^{2a^2} \frac{\exp(-\frac{1}{2}\omega)d\omega}{\omega\sqrt{(\omega - a^2)}}.$$

4887. *Proposed by Z. A. Melzak, McGill University*

What is the shortest length of infinitely strong, infinitely thin and perfectly inextensible string with which a smooth sphere of radius 1 can be tied up so that the resulting parcel can be picked up and carried by the string without danger of slipping apart?

4888. *Proposed by Emil Grosswald, University of Pennsylvania*

Determine the values of α, for which the following series converge:

$$S_1(\alpha) = \sum_{m=0}^{\infty} \frac{(m+1)^{m-1}}{m!}\alpha^m, \qquad S_2(\alpha) = \sum_{m=0}^{\infty} \frac{(m+1)^{m-1}}{m!}\alpha^{m+1}.$$

Show also that for those values of α, $S_2 = \log S_1$.

4889. *Proposed by M. S. Klamkin, AVCO Research, Wilmington, Mass.*

If k points are distributed at random at the vertices of a regular n-gon, determine the probability that the center of gravity of the k masses lies in a circle of radius r about the center of the n-gon. What does the probability function reduce to when $n \to \infty$?

4890. *Proposed by H. S. Shapiro, New York University*

Let $f(t)$ be positive, integrable and strictly decreasing in $(0, \infty)$, and suppose that the Fourier cosine transform of f is strictly decreasing in $(0, \infty)$. Then,

$$\int_0^{\infty} f(t)e^{izt}dt$$

is regular and schlicht in the upper half-plane.

4891. *Proposed by I. S. Gál, Cornell University*

Let X be a regular topological space and let A be a dense subset of X such that every net with values in A has a non-void adherence in X. Show that X is compact. Is regularity necessity?

4892. *Proposed by Andrew Kraus, Boulder, Colorado*

Show that if p is an odd prime and $k \geq 1$, then

$$(p^{k-1}+1)\cdots(p^k-1)+p^{p^{k-1}-k} \equiv 0 \pmod{p^{p^{k-1}}}.$$

Note that for $k=1$ this is Wilson's theorem.

SOLUTIONS

Linear Independence

4797 [1958, 452; 1959, 730]. *Proposed by D. J. Newman, Brown University.*

Prove that all expressions like $7\sqrt{19}/4-3\sqrt{7}+8\sqrt{6}/5$ are irrational. More specifically, prove that the square roots of the square-free integers are linearly independent over the rationals.

III. *Solution by Harley Flanders, University of California, Berkeley.* We use the rudiments of field theory to prove the theorem:

$\mathbf{T}_n$: *Let $a_1, \cdots, a_n$ be positive integers, coprime in pairs, no one of which is a perfect square. Then the 2^n algebraic integers*

$$(a_1^{e_1}\cdots a_n^{e_n})^{1/2}, \qquad e_j = 0, 1,$$

are linearly independent over the field Q of rationals.

$\mathbf{T}_1$ is true and we attack $\mathbf{T}_n$ where $n \geq 2$; and we have the induction hypothesis for smaller n. We define fields $K_0=Q$, $K_n=K_{n-1}(\sqrt{a_n})$. Then $[K_n: K_{n-1}] \leq 2$, hence $[K_n: Q] \leq 2^n$. Also 1, $\sqrt{a_n}$ span K_n linearly over K_{n-1}, hence the 2^n numbers above span K_n over Q. To prove they are linearly independent, hence a basis, we must prove $[K_n: Q]=2^n$. But by induction $[K_{n-1}: Q]=2^{n-1}$, so we must merely prove $[K_n: K_{n-1}]=2$. If not, then $K_n=K_{n-1}$, $\sqrt{a_n} \in K_{n-1}$. We have

$$\sqrt{a_n} = b + c\sqrt{a_{n-1}}, \qquad b, c \in K_{n-2}.$$

Thus $a_n=b^2+c^2a_{n-1}+2bc\sqrt{a_{n-1}}$. Since $\sqrt{a_{n-1}} \notin K_{n-2}$, we have $bc=0$.

Case 1. $c=0$. Then $\sqrt{a_n} \in K_{n-2}$ which contradicts $\mathbf{T}_{n-1}$ applied to $a_1, \cdots, a_{n-2}, a_n$.

Case 2. $b=0$. Then

$$\sqrt{a_n} = c\sqrt{a_{n-1}}, \qquad \sqrt{(a_{n-1}a_n)} = a_{n-1}c \in K_{n-2}.$$

This again contradicts $\mathbf{T}_{n-1}$ for the numbers $a_1, \cdots, a_{n-2}, a_{n-1}a_n$.

This completes the proof. The only real fact from field theory which we have used is the relation for linear dimension $[K: L][L: F]=[K: F]$ where $K \supset L \supset F$.

Also solved by Melvin Hausner, and Eugene Levine.

Radius of Convergence

4840 [1959, 317]. *Proposed by P. T. Bateman, University of Illinois*

Suppose $a_0>0$, $a_1 \geqq 0$, $a_2 \geqq 0, \cdots$, and put $A_n=a_0+a_1+\cdots+a_n$. Show that if as $n \to \infty$, $A_n \to \infty$ and $a_n/A_n \to 0$, then $\sum a_n x^n$ has radius of convergence unity.

Solution by Yi Chang, Fenn College, Cleveland, Ohio. Let r and R be the radii of convergence of $\sum a_n x^n$ and of $\sum A_n x^n$, respectively. Since $\lim A_n=\infty$, $\sum a_n x^n$ diverges for $x=1$, whence $r \leqq 1$.

Now $\lim a_n/A_n=0$ $(a_n \geqq 0)$ implies $r \geqq R$. It also implies $1=\lim A_n/A_n$ $=\lim (A_{n-1}+a_n)/A_n=\lim A_{n-1}/A_n$, whence $R=1$. Therefore $r=1$, as required.

Also solved by R. C. Bollinger, N. J. Fine, Leopold Flatto, L. R. Ford, Jr., Paul Gordon, Fritz Herzog, Joseph Lehner, A. E. Livingston, Lee Lorch, M. D. Mavinkure, A. A. Mullin, John Rainwater, James Reill, Peter Renz, J. E. Vinson, R. J. Whitley, Albert Wilansky, David Zeitlin, and the proposer. E. E. Kohlbecker finds the essential part of the result in Hardy, *Divergent Series*, p. 65.

A Set Everywhere Dense in the Plane

4843 [1959, 317]. *Proposed by Leopold Flatto, Brooklyn Polytechnic Institute*

Let S be a set everywhere dense in the plane. Does it follow that S is dense on some line segment in the plane?

Solution by Leo Moser, University of Alberta. No. In fact we construct a set S, dense in the plane, which contains no three collinear points. Let $Z_1, Z_2, \cdots$ be an enumeration of the algebraic points in the complex plane. We now let S be the points $S_1, S_2, \cdots$ defined as follows: $S_1=Z_1$, $S_2=Z_2$, while for $n>2$, $S_n=Z_n+2^{-n}e^{2\pi i/f(n)}$ where $f(n)$ is the smallest positive integer for which S_n is not collinear with any two of $S_1, S_2, \cdots, S_{n-1}$. Since we have infinitely many concyclic points to pick from and only a finite number of lines to avoid, $f(n)$ will certainly exist. The set S will have no three collinear points but since the Z's are dense and $|S_n-Z_n|=2^{-n}$ the set S will be dense.

Also solved by Donald Coleman, N. J. Fine, Fred Galvin, Fritz Herzog, Arthur Rosenthal, and the proposer.

Editorial Note. Coleman notes that Sierpinski has proved (*Fund. Math.*, vol. 1, 1920, pp. 112–115) that there exists a (non-measurable) subset of the plane which meets every closed set of positive measure, but which intersects any line in at most two points.

Rosenthal has shown (*S.-B. Bayer. Akad. Wiss.*, 1922, pp. 221–240—in particular p. 227) the existence of sets S everywhere dense in the plane which on every straight line have exactly 2 (or more generally, n) points.

The Ring of Polynomials in the Partial Differential Operators

4844 [1959, 318]. *Proposed by D. S. Kahn, Princeton University*

Consider the ring Q of polynomials in the partial differential operators $\partial/\partial x_1, \cdots, \partial/\partial x_n$ over the field of rational functions of $x_1, \cdots, x_n$. (1) Determine all (two-sided) ideals of Q. (2) Prove that Q is a primitive ring.

Solution by the proposer. In the case of two variables, we prove that the only ideals are (O) and Q.

A term $a_{mn}D_x^m D_y^n$ is said to have degree $m+n$. A polynomial is said to have deg p if p is the degree of the term of highest degree with non-zero coefficients. Let $\mathfrak{A} \neq 0$ be an ideal in Q and let A have least degree among non-zero elements of $\mathfrak{A}$. If deg $A = 0$, we are done. Assume deg $A = p > 0$. Then

$$(xy)^{-1}(A)(xy) = \sum_{i=0}^{p} a_{i,p-i} D_x^i D_y^{p-i} + (\text{terms of deg} < p) \in \mathfrak{A}.$$

Therefore $\overline{A} = A - (xy)^{-1}(A)(xy) \in \mathfrak{A}$ and $\overline{A} \neq 0$ with deg $< p$. This contradicts the minimality of deg A, whence $\mathfrak{A} = Q$.

Finally it is known that any simple ring with identity is primitive.

The same method of proof holds for n variables.

A Maximum Related to Orthogonal Matrices

4845 [1959, 426]. *Proposed by Fulton Koehler, University of Minnesota*

Let α be a real square matrix $[a_{ij}]$ and let $D(\alpha) = \sum_{i \geq j} a_{ij}$. Find the maximum of $D(\alpha)$ for the group of orthogonal matrices of given order n; and show that, as $n \to \infty$, the maximum is asymptotic to $(n/\pi) \log n$.

Solution by the proposer. For given n, the quantity $D(\alpha)$ is a continuous function of the n^2 elements a_{ij}, and the manifold of points (a_{ij}) corresponding to orthogonal matrices is compact; hence, $D(\alpha)$ has a maximum. Let $D(\alpha)$ assume the maximum for $\alpha = \gamma = [c_{ij}]$.

Let $\tilde{\gamma}(\theta)$ be the orthogonal matrix whose kth column is composed of the elements $c_{ik} \cos\theta + c_{im} \sin\theta$ $(i = 1, \cdots, n)$, whose mth column $(k \neq m)$ is composed of the elements $c_{im} \cos\theta - c_{ik} \sin\theta$ $(i = 1, \cdots, n)$, and whose other columns are identical with those of γ. Then

$$D(\tilde{\gamma}) - D(\gamma) = \sum_{i=k}^{n} \{c_{ik}(\cos\theta - 1) + c_{im} \sin\theta\} + \sum_{i=m}^{n} \{c_{im}(\cos\theta - 1) - c_{ik} \sin\theta\} \leqq 0$$

for all real θ. For infinitesimal θ this can be written

$$\left\{ \sum_{i=k}^{n} c_{im} - \sum_{i=m}^{n} c_{ik} \right\} \theta + O(\theta^2) \leqq 0$$

which is possible only if

$$\sum_{i=k}^{n} c_{im} - \sum_{i=m}^{n} c_{ik} \qquad (m, k = 1, \cdots, n). \tag{1}$$

Let $s_{km} = \sum_{i=k}^{n} c_{im}$, $\sigma = [s_{km}]$. Then σ is symmetric, and

$$D(\gamma) = s_{11} + \cdots + s_{nn} = \lambda_1 + \cdots + \lambda_n, \tag{2}$$

where $\lambda_1, \lambda_2, \cdots, \lambda_n$ are the characteristic values of σ. From the orthogonality of γ and the symmetry of σ we have

$$\sigma^2 = \sigma\sigma^* = \begin{bmatrix} n & n-1 & n-2 & \cdots & 1 \\ n-1 & n-1 & n-2 & \cdots & 1 \\ n-2 & n-2 & n-2 & \cdots & 1 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 1 & 1 & 1 & \cdots & 1 \end{bmatrix}, \tag{3}$$

and it can seen by inspection that

$$(\sigma^2)^{-1} = \begin{bmatrix} 1 & -1 & 0 & \cdots & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & 0 & & -1 & 2 \end{bmatrix}. \tag{4}$$

Let $2-\lambda = 2\cos\theta$. Then

$$\text{Det}\,[(\sigma^2)^{-1} - \lambda E] = (2\cos\theta - 1)\frac{\sin n\theta}{\sin\theta} - \frac{\sin(n-1)\theta}{\sin\theta} = \frac{\sin(n+1)\theta - \sin n\theta}{\sin\theta}$$

from the well-known formula

$$\begin{vmatrix} 2\cos\theta & -1 & 0 & \cdots & 0 \\ -1 & 2\cos\theta & -1 & \cdots & 0 \\ 0 & -1 & 2\cos\theta & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & 2\cos\theta \end{vmatrix} = \frac{\sin(m+1)\theta}{\sin\theta},$$

where m is the order of the determinant on the left. Hence the characteristic values of $(\sigma^2)^{-1}$ are $2(1-\cos r\pi)$, where $r = (2k-1)/(2n+1)$ and $k = 1, \cdots, n$. The characteristic values of σ are therefore $\pm(1-\cos r\pi)^{-1/2}/\sqrt{2}$, and

$$D(\gamma) \leqq (1/\sqrt{2}) \sum_{k=1}^{n} (1 - \cos r\pi)^{-1/2}. \tag{5}$$

We now show that equality must hold in (5). Let a matrix A of order n be defined by

$$A = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 1 & \cdots & 1 \\ 0 & 0 & 1 & \cdots & 1 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}.$$

Then $\sigma = A\gamma$. Let U be the orthogonal matrix such that $U\sigma^2U^*$ is the diagonal matrix whose kth diagonal element is $\frac{1}{2}(1-\cos r\pi)^{-1}$. Let δ be the diagonal matrix of order n whose kth diagonal element is $(1-\cos r\pi)^{-1/2}/\sqrt{2}$, and let $\gamma_1 = A^{-1}U^*\delta U$. Then $\gamma_1\gamma_1^* = A^{-1}U^*\delta^2U(A^{-1})^* = A^{-1}\sigma^2(A^{-1})^* = A^{-1}(A\gamma)(\gamma^*A^*)(A^{-1})^* = \gamma\gamma^* = E$, hence γ_1 is orthogonal. Also $D(\gamma_1)$ is the sum of the diagonal elements of $A\gamma_1 = U^*\delta U$ which is equal to

$$\sum_{k=1}^{n} (1/\sqrt{2})(1 - \cos r\pi)^{-1/2}.$$

Hence

$$D(\gamma) \geq D(\gamma_1) = \sum_{k=1}^{n} (1/\sqrt{2})(1 - \cos r\pi)^{-1/2}. \tag{6}$$

For given n, the required maximum is therefore

$$\sum_{k=1}^{n} \frac{1}{\sqrt{2}} (1 - \cos r\pi)^{-1/2} = \sum_{k=1}^{n} (2 \sin \tfrac{1}{2}r\pi)^{-1}$$

$$= \sum_{k=1}^{n} 1/r\pi + \sum_{k=1}^{n} \left\{(2 \sin \tfrac{1}{2}r\pi)^{-1} - (r\pi)^{-1}\right\} = (n/\pi) \log n + O(n),$$

since $\sum_{k=1}^{n} 1/(2k-1) = \frac{1}{2} \log n + O(1)$ and $(2 \sin \frac{1}{2}\theta)^{-1} - \theta^{-1}$ is bounded for $0 < \theta < \pi$.

Hermitian Matrices

4846 [1959, 427]. *Proposed by Olga Taussky, California Institute of Technology*

Show that a matrix which is similar to a real diagonal matrix is the product of two hermitian matrices one of which is positive definite. (The converse is known.)

Solution by W. V. Parker, Alabama Polytechnic Institute. If matrix A is similar to the real diagonal matrix D, then there exists a non-singular $n \times n$ matrix P such that $PAP^{-1} = D$. We may write $P = UH$, where U is a unitary matrix and

H is positive definite hermitian. Then $UHAH^{-1}U^* = D$ and $HAH^{-1} = H_1$ where H_1 is hermitian. It follows that $A = H^{-1}H_1H^{-1}H^2 = (H^{-1})^2HH_1H$, where $H^{-1}H_1H^{-1}$ and HH_1H are hermitian, and H^2 and $(H^{-1})^2$ are positive definite hermitian.

Also solved by A. C. Aitken, Wallace Givens, P. C. Greiner, Arthur F. Kaupe, Hans Schneider, H. Schwerdtfeger, and the proposer.

RECENT PUBLICATIONS

EDITED BY RICHARD V. ANDREE, University of Oklahoma

All books for review should be sent directly to R. V. Andree, Department of Mathematics, University of Oklahoma, Norman, Oklahoma, and not to any of the other editors or officers of the Association.

Principles and Applications of Random Noise Theory. By Julius S. Bendat. Wiley, New York, 1958. xxi+431 pp. $11.00.

While in this book mathematical methods are cleverly introduced as needed in the development of the physical theory, the mathematical portions are definitely limited to the mathematics necessary for immediate application. Included, however, as part of a chapter is an excellent but abbreviated course on probability. The relationship between the correlation functions of the statistician and power spectral density functions of the engineer is clearly delineated. The autocorrelation function $Ae^{-k|\tau|}\cos c\tau$ where A, k, and c are nonnegative constants is shown to be of considerable theoretical as well as experimental importance. The description, in some detail, of investigations of methods of controlling and evaluating this fundamental autocorrelation function is perhaps the primary concern of the book.

The author spells out his objectives in the preface and follows the table of contents with single paragraph summaries of each of the ten chapters. Following each chapter is a list of references, and finally following the last chapter is a quite complete bibliography of the generally accessible literature.

The author is to be congratulated for conceiving and carrying out so well the task of supplying such a well-connected and authoritative account of a virgin field, a field in which his own contributions have been most extensive.

HERBERT A. MEYER
University of Florida

Programming the IBM 650 Magnetic Drum Computer and Data-Processing Machine. By Richard V. Andree. Holt, New York, 1958. vi+114 pp. $2.95. (paper bound).

This book consists of a set of notes with many simple examples and exercises devoted to programming the IBM 650. The first four chapters are self-contained (71 pp.) giving the reader simple programming in machine language only, flow

charting, console operation, looping, stepping, testing, branching, scaling, examples of loss of accuracy, and use of subroutines. The remaining part of the book gives very brief discussions of the symbolic optimum assembly program, the Bell interpretive system, and the For Transit compiler with reference to standard manuals for more complete treatment.

Although very little use is made of symbolic programming, the book can be useful at any school with an IBM 650. Even if symbolic programming is introduced early by the instructor, the examples and exercises are easy to put into the symbolic form.

Also a minimum of the details of loading and general procedure is presented, thus making the text adaptable as introductory material for any IBM 650 computing center.

The author's friendly and informal style makes the text pleasant and easy to read.

DAVID B. DEKKER
University of Washington

Editorial Note. Since Professor Andree is the editor of this section, the editor-in-chief has assumed responsibility for editing this review.

The Theory of Groups. By Marshall Hall, Jr. Macmillan, New York, 1959. xiii+434 pp. $8.75.

This is a book which I wish I could put in the hands of every graduate student who has shown an interest in the elements of group theory. The first ten chapters would give him an excellent foundation in group theory, and there would still remain ten chapters for his delight. I will speak of the latter, some of which are unique.

Many mathematicians are vaguely aware that there is a famous problem for groups, called the Burnside problem. But what proportion, even if we restrict attention to algebraists alone, can state the problem, let alone distinguish it from the several forms of the restricted Burnside problem? This book supplies most of the tools currently used for the Burnside problem, proves many of the known theorems in the subject (with a few omissions where the only known proofs would severely try the patience of the reader) and brings the history of the problem as close to the present moment as is humanly possible. No other book goes half so far in this direction.

Who can direct a student to a book containing the theory of representations of a group of finite order n over a field of characteristic prime to n? This theory is contained *in principle* in H. Weyl's *The Classical Groups* (see footnote, p. 100) and has been given in many a graduate course, but Hall seems to be the first to publish a detailed account within the covers of a book. The advantages are numerous.

And, finally, there is no substitute in English for the eighty pages which Hall devotes to the connections between groups and projective planes. Car-

michael's *Theory of Groups of Finite Order*, which does something along this line, appeared before the existence of most of the present material.

To this reviewer, the book seems like a lively and attractive grandson of Burnside's *Theory of Groups of Finite Order*, a grandson, let it be added, with a marked personality of his own.

R. H. BRUCK
University of Wisconsin

Sampling Opinions. An Analysis of Survey Procedures. By Frederick F. Stephan and Philip J. McCarthy. Wiley, New York, 1958. 451 pp. $12.00.

From a pragmatic point of view, statistics is the art of making inferences about populations based upon samples from these populations. This book studies the problems of applying basic techniques of statistics to investigating human populations. The emphasis is on socio-economic phenomena, but there is much of interest for all scientific investigators. The book is written in a nonmathematical fashion, but is unlike most statistical cookbooks in two important respects: a) It reflects a sophisticated training in modern theoretical statistics on the part of the authors, and b) it should be of considerable interest to an audience with such training. To elaborate on the second point, I question that much of the book will make sense to one not versed in statistics, despite its nonmathematical appearance.

MEYER DWASS
Northwestern University

Studies in Linear and Non-Linear Programming. Edited by Kenneth Arrow, Leonid Hurwicz, and Hirofumi Uzawa. Stanford University Press, Stanford, California, 1958. 229 pp. $7.50.

This book is the second in the Stanford Mathematical Studies in the Social Sciences and is a well-integrated collection of research papers that deal with the fields of linear and nonlinear programming. In addition to the contributions of the editors there are papers by Hollis B. Chenery, Selmer M. Johnson, Samuel Karlin, Thomas Marschak, and Robert M. Solow. Although the original problems studied were in the area of transportation, assignment, and storage, the theory as it is developed here may be applied to general problems involving the optimal or feasible allocation of limited resources.

In addition to an introductory first chapter, that also summarizes the main results, there are three parts. Part I presents the basic mathematical structure in the form of fundamental existence and duality theorems for linear, convex, and concave programming. In Part II, the gradient of a function is defined and applied to solve constrained maximization problems by means of successive approximations. Part III is devoted to applying and extending the techniques to problems in price speculation, machine assignment, and interindustry analysis.

Because it is primarily devoted to the exposition of general results, there

should be at least three types of users of this excellent book. First, it should appeal to those who are interested in research in mathematical programming and allocation theory. It should be of special use to those in mathematics, mathematical economics, and operations research who are concerned with solving industrial and economic policy problems. Finally, with the further addition of examples and problems, this book would prove to be an excellent basis for an advanced course in mathematical programming to be given to graduate students in the mathematical or management sciences. The only modification this reviewer would make would be to include material on programming with uncertainty.

BURTON V. DEAN
Case Institute of Technology

Introduction to Mathematical Analysis with Applications to Problems of Economics. By Paul H. Daus and William M. Whyburn. Addison-Wesley, Reading, Mass., 1958. 244 pp. $6.50.

This book covers such material as elementary notions of analytic geometry, differentiation and integration, partial differentiation, maxima and minima including constraint conditions and the method of Lagrange multipliers, curve fitting (straight line, parabola, exponential and power, laws by method of least squares), correlation. Spread throughout these mathematical topics are the applications to economics, such as supply and demand curves, market equilibrium, effect of taxation, elasticity, marginal revenue, profit under monopoly, production functions, indifference maps, utility index, relative preference, etc. The material is well written and the large number of interesting problems are more than adequate for a one-semester course. Especially recommended for students of business and economics.

CLETUS OAKLEY
Haverford College

Methods in Numerical Analysis. By Kaj L. Nielsen. Macmillan, New York, 1957. xiii+382 pp. $7.25.

Since this text appeared, the reviewer has used it in two long sessions and in two summer sessions for a beginning course for technical students pursuing studies in numerical analysis. The reviewer has had opportunity to examine the first printing (1956) and the second printing (1957). A number of harmless and/or annoying misprints in the first printing were corrected in the second printing.

Many teachers prefer to use a textbook with errors on the theory that a careful students will learn much more by discovering an error and reconstructing the material in correct fashion. Perhaps the reviewer is such a teacher, for there are numerous errors in Nielsen's book, and yet he has continued to use Nielsen's book.

The reviewer finds no basis in fact for the statement on page 42: "If the nth

differences of a function tabulated at equally spaced intervals are constant, the function is a polynomial of degree n."

The answer list is poorly prepared. For example, Exercise 7a, page 168 uses five significant digit tabular data, and the answers are six significant digit numbers.

In many places the author writes that "to a first order approximation $\Delta N = \sum_{i=1}^{n} \Delta x_i (\partial f/\partial x_i)$" or some similar statement. This reviewer instructs all his students to replace the equality sign above by some symbol to indicate approximate equality, or order of magnitude of equality, or the use of a phrase like leading error terms.

Nielsen's style of writing appeals to many technical students who are not majoring in mathematics. If the class is taught by a mathematician who will make sure that the pupils discover and weed out the errors in the book, this text can be used with profit.

ROBERT E. GREENWOOD
The University of Texas

Linear Programming: Methods and Applications. By Saul I. Gass. McGraw-Hill, New York, 1958. xii+233 pp. $6.75.

This is the first textbook on linear programming which gives a comprehensive treatment of the subject and is suitable for use in a college mathematics course. The covering is remarkably complete, including an adequate survey of matrices, vectors, convex sets and linear inequalities, a detailed description of the theoretical and computational aspects of methods of solution, and applications. It is the author's intent that the text be appropriate for a one-semester course at the senior or first-year-graduate level. Such a course would have great value for students of many disciplines, and for this purpose the book warrants unreserved recommendation.

After an introductory chapter in which the general linear programming problem is stated and several examples are given, there follows a compact discussion of essential mathematical background (matrices, convex sets, etc.). Chapter 3 treats properties of solutions to linear programming problems, especially extreme point solutions. Chapter 4 describes the simplex method and Chapter 5 deals with duality in linear programming. Recent and/or special methods are covered in the next four chapters (the revised simplex method, degeneracy procedures, parametric linear programming, additional computational techniques.)

The remaining three chapters are concerned with applications. No attempt has been made to encompass all of these, but a good variety of representative examples is covered. The transportation problem and its more popular variants are studied in detail. Unfortunately, the max-flow, min-cut method of Ford and Fulkerson receives only passing mention. One might also wish that at least one example of a nonlinear problem solvable by piecewise linear approximation to

the objective form had been included. The final chapter treats the equivalence of the zero-sum two-person game and the linear programming problem.

Each chapter is followed by a short set of exercises suitable for pencil-and-paper solution, and the book has been written so that recourse to a computer is not required. The exercises achieve their purpose, but this reviewer would have appreciated more "motivated" problems. A comprehensive bibliography is appended, containing 102 general references and 91 items arranged by area of application.

L. E. WARD, JR.
University of Oregon

Elementary Algebra. By D. S. Russell, Allyn and Bacon, Boston, 1959. ix+297 pp. $4.50.

The author introduces this text as "designed . . . for the college student who has had no previous training in algebra as well as for . . . an intensive review of the basic fundamentals." The presentation is, however, as well suited for high school as it is for college. The author states that "the theory is presented in a simple style." This is surely true, although at times one might wish for a less intuitive approach, particularly in a text for college use. In some places there is an unfortunate lack of precision of statement. For examples: "a^3 means that the number represented by a is multiplied together three times"; (How does one multiply a number together?), and "a *real* number is any number that can be associated with a point on a line." The associative, commutative and distributive laws could have been well introduced where they are first used in formulas, rather than later. Numerical multiplication by use of binomial factors is treated in a short section. This is an interesting inclusion often omitted in other texts. In the section on quadratics the discriminant is not mentioned as such, although some exercises provide opportunity for its use. Problems are numerous and generally well chosen although more on rates and distances, mixtures, and fractions might have been included in the "stated problems." "Review tests" included seem to have worthwhile possibilities as teaching aids. This is an adequate text but has little to recommend it over some others on the market.

JAMES L. SIMPSON
Montana State College

Calculus. By R. E. Johnson and F. L. Kiokemeister. Allyn and Bacon, Boston, 1959. xii+634 pp. $7.75.

This text is designed for students who have had algebra, trigonometry, plane geometry, and analytic geometry. It contains applications of the calculus with an excellent discussion of its rigorous basis. Illustrative examples are used to motivate new concepts in advance of their theoretical discussion.

The definite integral is carefully handled with a discussion of upper and lower integrals, and the integral as the limit of a sequence of Riemann sums.

There is a chapter on the basic properties of continuous and differentiable

functions. This chapter contains an excellent discussion of approximation by Taylor polynomials which should be of real help to the students in handling Taylor series and its remainder later.

Convergence of infinite series is handled with more than usual care, and proofs are given for termwise differentiation and integration of power series. Solid analytic geometry is treated both with and without vectors. One whole chapter and part of another are devoted to vectors.

Second order linear differential equations with constant coefficients are stressed in the final chapter. Material in this chapter and on vectors will help the engineering student whose mathematics lags behind his need for it in other courses.

The text contains all the traditional calculus material and should be readily adaptable to a variety of courses.

Finally there are collections of facts and formulas from trigonometry and analytics, a table of integrals, various numerical tables, and answers to odd numbered problems.

The book set up is attractive, and the material well presented. The reviewer recommends the text as a worthy continuation of the high standard set by the book *Calculus with Analytic Geometry* by the same authors.

W. N. Huff
University of Oklahoma

Linear Operators, Part I: General Theory, by N. Dunford and J. T. Schwartz. Interscience, New York, 1958. xiv+858 pp. $23.00.

The present volume is the first of a two-volume work which is designed to give a self-contained and comprehensive survey of linear analysis. The first volume is primarily devoted to a study of Banach spaces and the linear transformations between such spaces. In addition to a wealth of material concerning the theory of linear operators, one finds many special features worthy of note. One such is an extensive treatment of the theory of integration in a form which applies to both vector-valued and extended real-valued measures. Another is a detailed comparison of the properties of 28 special spaces. Complete proofs are presented of all propositions except those within the exercise lists. The latter, incidentally, are distributed throughout the volume and contain a multitude of the applications of the theory to classical analysis. Each chapter includes notes of a historical nature and notes concerning the differing directions in which the theory has developed. The volume closes with a list of references 96 pages in length and covering the material of both the current and prospective volumes.

This volume should be an invaluable reference work, both from the point of view of the worker in the field of linear analysis and from that of a teacher who is presenting some aspects of this theory. The mathematical world will surely find this volume a most useful one for a considerable time to come.

Paul Civin
University of Oregon

NEWS AND NOTICES

EDITED BY LLOYD J. MONTZINGO, JR., University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to L. J. Montzingo, Jr., Mathematical Association of America, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

PERSONAL ITEMS

University of California, Riverside: Dr. R. L. McKinney has been promoted to Assistant Professor; Dr. Auguste Forge, Duke University, has been appointed Visiting Assistant Professor; Dr. R. E. Haymond, University of Oregon, has been appointed Instructor; Miss Dorothy I. Koehler, Cornwall College, Montego Bay, Jamaica, has been appointed Associate.

University of Florida: Assistant Professor Diran Sarafyan is on leave of absence to study in Rome, Italy, on a National Science Foundation Faculty Fellowship; Dr. L. R. Luckenbach has been appointed Int. Assistant Professor; Mrs. Joyce C. Cundiff, Alabama Polytechnic Institute, Mr. R. V. Gentry, Martin Company, Mr. J. W. Kenelly, and Mr. J. W. Meux have been appointed Instructors; Mr. R. E. Sellers, Radio Corporation of America, has been appointed Int. Instructor.

Eastern Michigan University: Assistant Professor Madeline Early has been promoted to Associate Professor; Mr. B. L. Goosey, Pennsylvania State University, has been appointed Assistant Professor.

Iowa State Teachers College: Associate Professor E. Glenadine Gibb has been promoted to Professor; Assistant Professor Augusta Schurrer has been promoted to Associate Professor; Mrs. Ina Mae Silvey has been promoted to Assistant Professor; Miss Ruth L. Erckmann, Central High School, Sioux City, Iowa, has been appointed Assistant Professor; Mr. R. L. Yount, Illinois Wesleyan University, and Mr. J. E. Bruha have been appointed Instructors.

Lafayette College: Dr. John Bender, Rutgers University, and Dr. J. S. Klein, Wilson College, have been appointed Associate Professors; Mr. John Robinson, Operations Research, Inc., Adelphi, Maryland, has been appointed Assistant Professor.

University of Louisville: Professor Guy Stevenson has retired as Head of the Department of Mathematics, but will continue as Professor of Mathematics; Associate Professor W. H. Spragens, Jr. has been promoted to Professor and Head of the Department of Mathematics.

University of Massachusetts: Miss Lorraine D. Lavallee, University of Michigan, Miss Carmela M. Landolfi, Ohio State University, Mr. R. M. Kennedy, and Mr. Y. S. Farsakh have been appointed Instructors.

Pennsylvania State University: Assistant Professor C. C. Faith is at Heidelberg University, Germany, on a NATO fellowship; Associate Professor R. G. Ayoub has a grant awarded by the American Mathematical Society and supported by the U. S. Air Force Office of Scientific Research, to study at Harvard University and Oxford University, England; Dr. R. P. Kanwal, Mathematics Research Center, University of Wisconsin, has been appointed Associate Professor; Dr. W. L. Harkness, Michigan State University, and Dr. D. H. Husemoller, University of Rochester, have been appointed Assistant Professors; Professor Beatrice L. Hagen has retired with the title Professor Emeritus.

Rutgers, The State University: Professor E. R. Ott, Chairman of the program in applied and mathematical statistics, has been named Director of the newly established Statistics Center; Dr. M. B. Wilk is Director of Research at the Center; Drs. R. S. Pinkham, M. E. Wescott, and H. F. Dodge, will serve on the staff of the Center.

Texas A. & M. College: Assistant Professors M. E. Tittle and S. A. Sims have been

promoted to Associate Professors; Mr. H. D. Perry has been promoted to Assistant Professor; Professor W. L. Porter has retired with the title Professor Emeritus; Mr. C. M. Pearcy has been appointed Assistant Professor; Mr. James Harris has been appointed Instructor; Mr. Peter Terwey, on leave from Lamar State College of Technology, has been appointed Visiting Assistant Professor.

University of Western Ontario: Assistant Professor William Wehlau has been promoted to Associate Professor; Dr. G. E. Cross, Victoria College, British Columbia, Canada, and Dr. J. Sanders, Glasgow University, Scotland, have been appointed Assistant Professors; Mr. R. M. Ellis, University of California, Los Angeles, has been appointed Instructor.

Wayne State University: Professor Sze-Tsen Hu was conferred the distinguished degree of Doctor of Science by the University of Manchester, England; Associate Professor O. G. Owens has been promoted to Professor; Assistant Professor Bess E. Allen has been promoted to Associate Professor; Dr. Franz Schnitzer has been promoted to Assistant Professor; Mr. Perry Scheinok, Indiana University, has been appointed Instructor.

Mr. J. F. Blackburn, I.B.M. Eastern Regional Office of the Data Processing Division, has joined the International Business Machines Research Center staff as Manager of Research Sales and Contracts at the Mohansic Laboratory in Yorktown, New York.

Professor A. T. Brauer, University of North Carolina, has been promoted to a Kenan Professorship.

Mr. Sol Broder, Brooklyn College, has accepted a position as Supervisor of Engineering Applications with Republic Aviation Corporation, Farmingdale, Long Island, New York.

Brother Joseph Heisler, C.S.C., University of Notre Dame, has been appointed Mathematics Teacher at the Archbishop Curley High School, Miami, Florida.

Mr. Frederick Brundage, University of Kansas, has accepted a position as Computer Analyst with the Military Products Division of the International Business Machines Corporation, Owego, New York.

Dr. E. W. Cheney, Convair-Astronautics, San Diego, California, has accepted a position as Member of the Technical Staff with Space Technology Laboratories, Inc., Los Angeles, California.

Mr. S. H. Coleman has been appointed Junior Instructor at the University of Virginia.

Mr. Demetrios Counes has accepted a position as Mathematician with the Lehigh Design Company, Newark, New Jersey.

Professor A. H. Diamond, Stevens Institute of Technology, has been appointed Professor at Webb Institute, Glen Cove, Long Island, New York.

Mr. Chester Domoracki has accepted a position as Methods Planner with the Western Electric Company.

Mrs. Jeanne C. Gardner, University of Pittsburgh, has been appointed Instructor at the University of Arkansas.

Mrs. Aiko M. Hormann, Westinghouse Electric Corporation, Pittsburgh, Pennsylvania, has accepted a position as Senior Analyst with the Systems Development Corporation, Santa Monica, California.

Mr. R. R. Hornby, University of Nebraska, has accepted a position as Associate Research Engineer with the Boeing Airplane Company, Wichita, Kansas.

Dr. J. E. Householder, Sr., University of Colorado, has been appointed Assistant Professor at Humboldt State College.

Mr. C. M. Hughey has accepted a position as Planning Engineer with the Western Electric Company, Inc., Winston-Salem, North Carolina.

Mr. R. L. Jacobsen, Stanford University, has been appointed Assistant Instructor at the State University of Iowa.

Mr. A. J. Kainen, Georgia Institute of Technology, has been appointed Assistant Professor at the University of Tampa.

Dr. Ferdinand Kertes has retired as Head of the Department of Mathematics at Perth Amboy High School, New Jersey.

Dr. R. K. Meany has been appointed Assistant Professor at Texas Christian University.

Mr. L. F. Nichols, Picatinny Arsenal, Dover, New Jersey, has been appointed Instructor at Stevens Institute of Technology.

Dr. Gordon Raisbeck has taken a one year leave of absence from the Bell Telephone Laboratories to accept a temporary appointment to the staff of the Advanced Research Projects Division of the Institute of Defense Analyses, Washington, D. C.

Mr. K. A. Retzer, Saunemin High School, Saunemin, Illinois, has been appointed Assistant Professor at Illinois State Normal University.

Mr. Winston Riley, III, International Business Machines Corporation, Cleveland, Ohio, has joined the Operations Research Branch of the Corporation for Economic and Industrial Research, Arlington, Virginia.

Mr. J. F. H. Schluep, Cato Meridian Central School, Cato, New York, has been appointed Associate Professor at the State University of New York, College of Education at Oswego.

Mr. M. C. Schwartz, Philco Corporation, has been appointed Lecturer at Harpur College of the State University of New York.

Mr. E. R. Willard, University of Rochester, has been appointed Instructor at Hamilton College.

Professor Emeritus Joseph Bowden, Adelphi College, died in June, 1959. He was a charter member of the Association.

Professor Emeritus Arthur Rosenthal, Purdue University, died September 8, 1959. He was a member of the Association for eighteen years.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

THE OCTOBER MEETING OF THE INDIANA SECTION

The annual fall meeting of the Indiana Section of the Mathematical Association of America was held on October 30, 1959, at Butler University, Indianapolis, Indiana, in conjunction with the meeting of the Indiana Academy of Science. Professor G. N. Wollan, Chairman of the Mathematics Section of the Academy presided at the morning session and Professor K. H. Carlson, Chairman of the Indiana Section of the Association presided at the afternoon session.

Professor Carlson was elected Chairman of the Mathematics Section of the Academy for 1960.

The following program was presented:

Morning Session—10 A.M. to 1 P.M.

1. *Computers and the collegiate mathematics program*, a symposium, by Professor P. T. Mielke, Wabash College, Professor Seymour Parter, Indiana University, Dr. K. L. Nielsen, Allison Division of General Motors, and Dr. John Lawrence, Chicago Office of I.B.M.

2. *Some advantages of Riemann-Stieltjes integration*, by Professor G. L. Krabbe, Purdue University.

3. *Uncertainty and entropy*, by Professor J. H. Abbott, Purdue University.

Afternoon session—3:30 P.M. to 5:30 P.M.

4. *Generalized functions—a survey*, by Professor Michael Golomb, Purdue University.

5. *A new introduction to the ideas and methods of trigonometry*, by Professor R. J. Thomas, DePauw University.

6. *A new look at least squares*, by Professor Richard Dowds, Butler University.

7. *A definition of polynomials on topological groups, and some examples*, by Professor F. W. Carroll, Purdue University.

C. F. BRUMFIEL, *Secretary*

THE OCTOBER MEETING OF THE OKLAHOMA SECTION

The fall meeting of the Oklahoma Section of the Mathematical Association of America was held at Oklahoma City University, Oklahoma, on October 23, 1959. Professor E. P. Richardson, Chairman of the Section presided. There were 114 persons in attendance, including 64 members of the Association.

The following officers were elected for a one-year term: Chairman, Professor Katherine Mires, Northwestern State College; Vice-Chairman, Professor R. G. Laatsch, University of Tulsa; Secretary-Treasurer, Professor R. V. Andree, University of Oklahoma.

The fall meeting of the Oklahoma Section is held in conjunction with the Oklahoma Education Association and is devoted to expository papers of particular interest to high school teachers. Research papers are presented in the spring meeting. The following papers were presented:

1. *Mann's theorem on the Landau-Schnirlmann alpha plus beta hypothesis*, by Professor Charles Nicol, University of Oklahoma.

2. *Probability and inductive inference*, by Dr. F. A. Graybill, Oklahoma State University of Agriculture and Applied Science.

3. *Recent developments in geometry*, by Professor T. K. Pan, University of Oklahoma.

4. *Metric postulates for plane geometry*, by Professor Gene Levy, University of Oklahoma.

5. *The relation between abstract algebra and the high school mathematics curriculum*, by Professor J. E. Hoffman, Oklahoma State University of Agriculture and Applied Science.

6. *Vectors in high school geometry*, by Professor Arthur Bernhart, University of Oklahoma.

7. *Experience with the UICSM project*, by Professor Eunice Lewis, University of Oklahoma.

8. The *luncheon speaker* was Professor Jack Forbes of Ball State Teachers' College, Muncie, Indiana. Professor Forbes spoke to the combined membership of the Oklahoma Section of the Mathematical Association of America and the Oklahoma Council of Teachers of Mathematics.

9. Another combined session of these two groups was a *panel discussion* led by three of the teachers who participated as members of the writing group for secondary school text materials which met at Boulder, Colorado, during the summer of 1959.

Committee reports included the report of the high-school lecture committee, which is serving as a clearing house for invitations from high school mathematics groups desiring college speakers. Finances will not permit sponsoring a regular traveling lecturer, but the

committee has served as a clearing house for information concerning available speakers.

The high school contest committee reports that 57 schools participated in the contest during the spring of 1959, and that Booker T. Washington High School of Idabel, Oklahoma, had the highest team score in Region 8, which includes Wyoming, Utah, Colorado, Oklahoma, Texas, New Mexico, and Arizona. This school placed four students on the Region 8 Honor Roll.

The touring lectureship committee reported that a proposal had been submitted jointly with the Oklahoma and Arkansas Academies of Science to sponsor visiting lectures in mathematics in Oklahoma and Arkansas. Later, the proposal was withdrawn on the grounds that it was too late in the year to secure lecturers of the quality desired who could devote full time to the project. The committee reports they plan to submit another proposal and hope it will receive earlier action.

R. V. ANDREE, *Secretary*

CALENDAR OF FUTURE MEETINGS

Forty-first Summer Meeting, Michigan State University, East Lansing, Michigan, August 29–September 1, 1960.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

Forty-fourth Annual Meeting, Willard Hotel, Washington, D. C., January 25–27, 1961.

ALLEGHENY MOUNTAIN, Grove City College, Grove City, Pennsylvania, April 30, 1960.

ILLINOIS, Illinois Wesleyan University, Bloomington, May 13–14, 1960.

INDIANA, Earlham College, Richmond, May 7, 1960.

IOWA, State University of Iowa, Iowa City, April 22, 1960.

KANSAS, Kansas State College of Pittsburg, April 30, 1960.

KENTUCKY, University of Kentucky, Lexington, April 30, 1960.

LOUISIANA-MISSISSIPPI, Buena Vista Hotel, Biloxi, Mississippi, February 19–20, 1960.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, University of Virginia, Charlottesville, May 7, 1960.

METROPOLITAN NEW YORK, City College, New York, April 2, 1960.

MICHIGAN, University of Michigan, Ann Arbor, March 26, 1960.

MINNESOTA, St. Olaf College, Northfield, May 7, 1960.

MISSOURI, Central Missouri State College, Warrensburg, April 30, 1960.

NEBRASKA, University of Nebraska, Lincoln, April 23, 1960.

NEW JERSEY

NORTHEASTERN

NORTHERN CALIFORNIA

OHIO, Kent State University, May 7, 1960.

OKLAHOMA, University of Oklahoma, Norman, April 15–16, 1960.

PACIFIC NORTHWEST, State University of Montana, Missoula, June 17, 1960.

PHILADELPHIA

ROCKY MOUNTAIN, United States Air Force Academy, Colorado Springs, May 6–7, 1960.

SOUTHEASTERN, University of South Carolina, Columbia, April 1–2, 1960

SOUTHERN CALIFORNIA, Los Angeles State College, March 12, 1960.

SOUTHWESTERN, Air Force Missile Development Center, Holloman Air Force Base, New Mexico, April 8–9, 1960.

TEXAS, San Antonio College, April 8–9, 1960.

UPPER NEW YORK STATE, University of Rochester, May 7, 1960.

WISCONSIN, Mount Mary College, Milwaukee, May 7, 1960.

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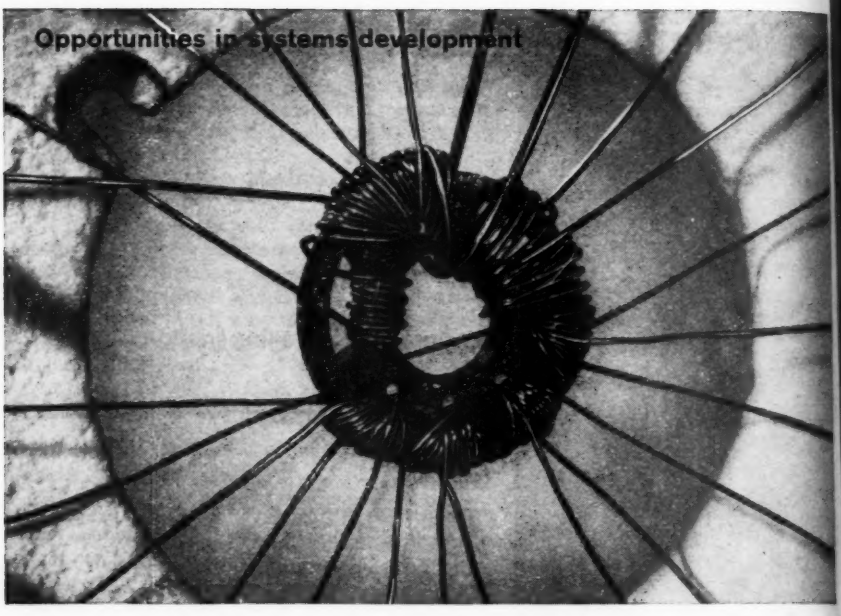

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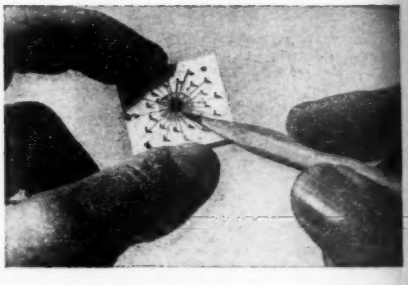

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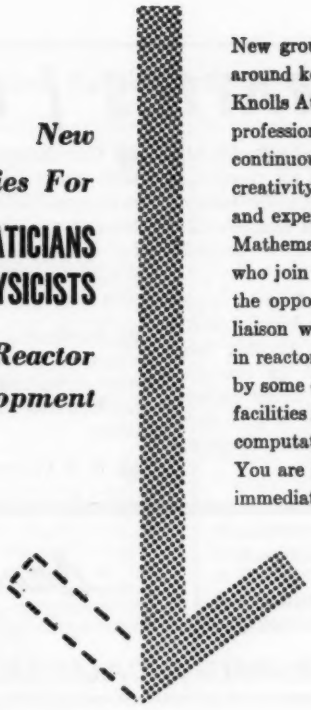

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